VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2018

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

ANS

STUDENT NUMBER | | | | | | | | | Letter

MATHEMATICAL METHODS

Written examination 1

Wednesday 7 November 2018

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 14 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (3 marks)

a. If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$. 1 mark

$\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2 \times (-9x^2 + 2x)$.

b. Let $f(x) = \frac{e^x}{\cos(x)}$.

Evaluate $f'(\pi)$. 2 marks

$f'(x) = \frac{e^x \cos(x) - e^x \times -\sin x}{(\cos x)^2}$

$f'(\pi) = \frac{e^\pi \cos \pi - e^\pi \times -\sin \pi}{(\cos \pi)^2}$

$= \frac{e^\pi \times -1 - e^\pi \times 0}{(-1)^2}$

$= -e^\pi$

Question 2 (3 marks)

The derivative with respect to x of the function $f: (1, \infty) \to R$ has the rule $f'(x) = \frac{1}{2} - \frac{1}{(2x-2)}$.

Given that $f(2) = 0$, find $f(x)$ in terms of x.

$f(x) = \int \frac{1}{2} - \frac{1}{2x-2}\, dx$

$f(x) = \frac{1}{2}x - \frac{\log_e(2x-2)}{2} + c.$

$0 = \frac{1}{2}\times 2 - \frac{\log_e(4-2)}{2} + c.$

$0 = 1 - \frac{1}{2}\log_e 2 + c$

$c = \frac{1}{2}\log_e 2 - 1 \rightarrow f(x) =$ ~~$\frac{x}{2} + \frac{1}{2}\log_e\left|\frac{2}{2x-2}\right|$~~

$= \frac{x}{2} - \frac{\log_e(2x-2)}{2} + \frac{\log_e 2}{2} - 1$

$= \frac{1}{2}\left(x + \log_e\left|\frac{2}{2x-2}\right| - 2\right)$

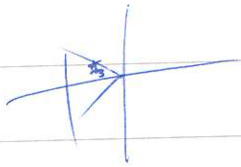
Question 3 (5 marks)

Let $f: [0, 2\pi] \to R,\ f(x) = 2\cos(x) + 1$.

a. Solve the equation $2\cos(x) + 1 = 0$ for $0 \le x \le 2\pi$. 2 marks

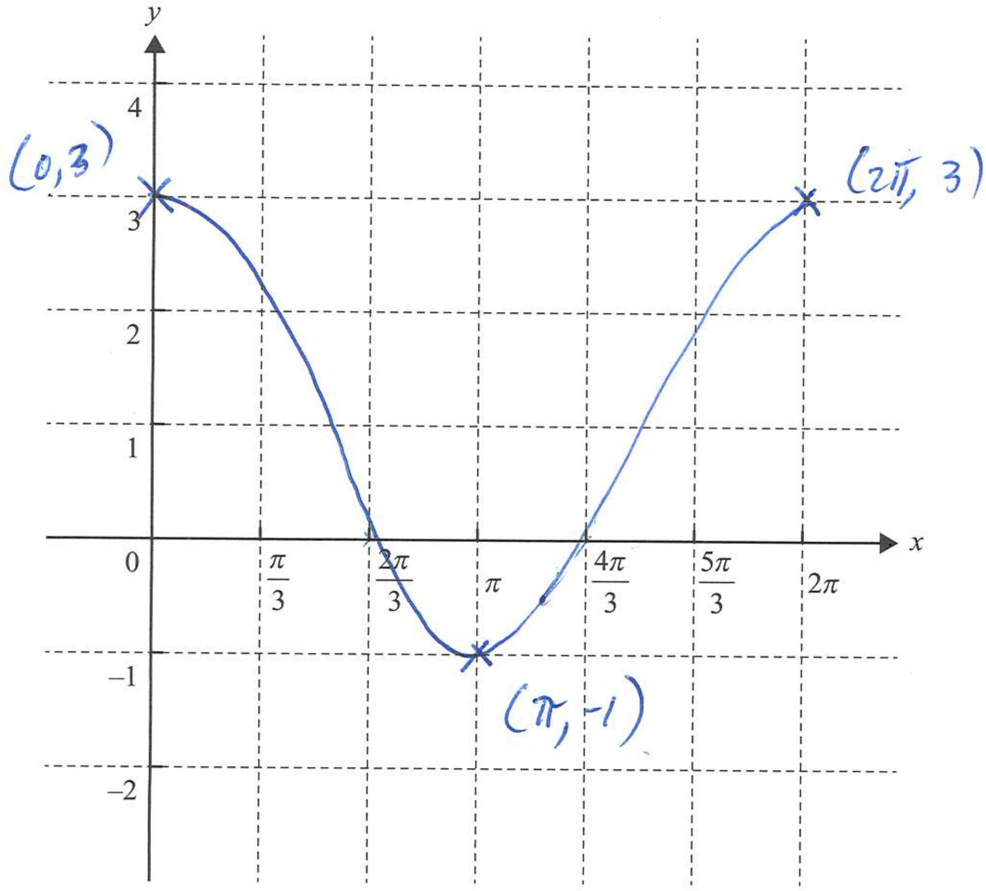
$\cos x = -\frac{1}{2}.$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}.$



b. Sketch the graph of the function f on the axes below. Label the endpoints and local minimum point with their coordinates.

3 marks



Question 4 (2 marks)

Let X be a normally distributed random variable with a mean of 6 and a variance of 4. Let Z be a random variable with the standard normal distribution.

a. Find $\Pr(X > 6)$. 1 mark

$\Pr(X > 6) = 0.5$.

b. Find b such that $\Pr(X > 7) = \Pr(Z < b)$. 1 mark

$\sigma^2 = 4 \rightarrow \sigma = 2$.

$z = \dfrac{x - \mu}{\sigma}$

Symmetry

$\Pr(X > 7) = \Pr(X < 5)$.

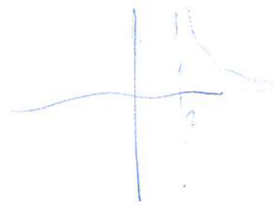
$z = \dfrac{5-6}{2} = -\dfrac{1}{4}$

Question 5 (3 marks)

Let $f: (2, \infty) \rightarrow R$, where $f(x) = \dfrac{1}{(x-2)^2}$.

State the rule and domain of f^{-1}.

$(0, \infty)$.



$x = \dfrac{1}{(y-2)^2}$

$\sqrt{x} = \dfrac{1}{y-2}$

$\dfrac{1}{\sqrt{x}} = y - 2$

$y = \dfrac{1}{\sqrt{x}} + 2$.

Domain $(0, \infty)$.

$f^{-1}(x) = \dfrac{1}{\sqrt{x}} + 2$

Question 6 (4 marks)

Two boxes each contain four stones that differ only in colour.

Box 1 contains four black stones.

Box 2 contains two black stones and two white stones.

A box is chosen randomly and one stone is drawn randomly from it.

Each box is equally likely to be chosen, as is each stone.

a. What is the probability that the randomly drawn stone is black? 2 marks

1 . B + 2 B

$\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{2}{4}$

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

b. It is not known from which box the stone has been drawn.

Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1? 2 marks

$Pr(1 | B) = \frac{1/2}{3/4}$

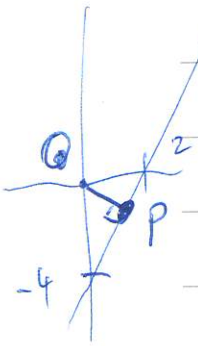
$= \frac{1}{2} \times \frac{4}{3}$

$= \frac{2}{3}$.

Question 7 (5 marks)

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

a. Find the coordinates of P. 3 marks



$P(x, y) = (x, 2x-4)$.

$m_{OP} = -\frac{1}{2} = \frac{y-0}{x-0}$

$-\frac{1}{2} = \frac{y}{x}$

$y = -\frac{1}{2}x$

$2x - 4 = -\frac{1}{2}x$

$\frac{5}{2}x = 4$

$x = \frac{2}{5} \times 4 = \frac{8}{5}$

$y = -\frac{1}{2} \times \frac{8}{5}$

$= -\frac{4}{5}$

$\left(\frac{8}{5}, -\frac{4}{5}\right)$.

b. Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers. 2 marks

$d_{OP} = \sqrt{\left(\frac{8}{5} - 0\right)^2 + \left(-\frac{4}{5} - 0\right)^2}$

$= \sqrt{\frac{64}{25} + \frac{16}{25}}$

$= \frac{\sqrt{80}}{5}$

$= \frac{4\sqrt{5}}{5}$

$80 = 8 \times 10$

$= 4 \times 20$

$= 4 \times 4 \times 5$

Question 8 (7 marks)

Let $f: R \to R$, $f(x) = x^2 e^{kx}$, where k is a positive real constant.

a. Show that $f'(x) = xe^{kx}(kx + 2)$. 1 mark

Let $u = x^2 \quad v = e^{kx}$

$u' = 2x \quad v' = ke^{kx}$

$f'(x) = x^2 ke^{kx} + 2xe^{kx}$

$= xe^{kx}(kx + 2).$

b. Find the value of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection. 2 marks

$x^2 e^{kx} = xe^{kx}(kx+2)$

$0 = kx^2 e^{kx} + 2xe^{kx} - x^2 e^{kx}$

$0 = xe^{kx}(kx + 2 - x)$

$x = 0 \quad e^{kx} = 0 \quad kx + 2 - x = 0$

No Sol^n. $\quad (k-1)x + 2 = 0$

$x = \frac{-2}{k-1}$

No Sol^n when $k = 1$

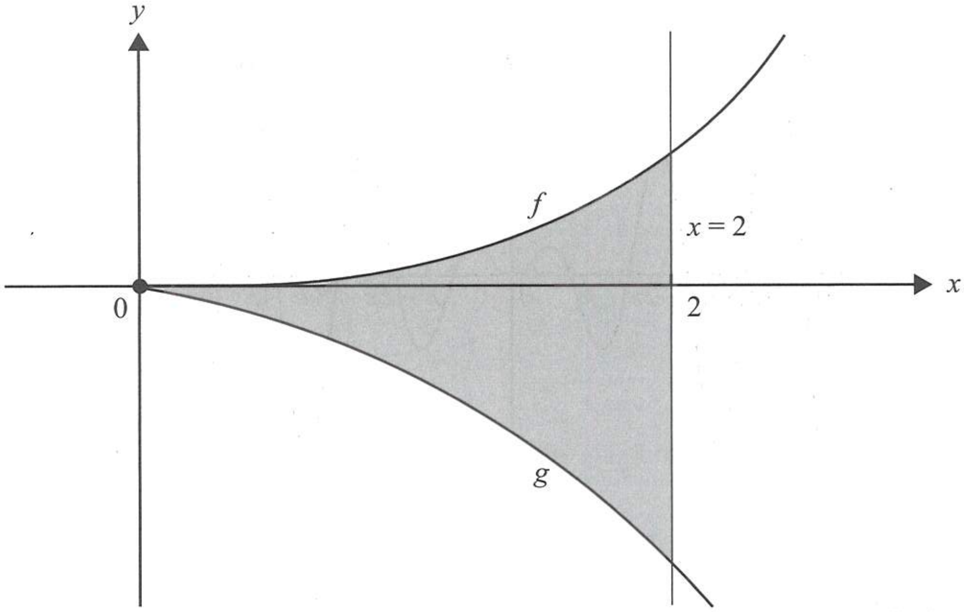
when $k = 1$.
only one Solution
$x = 0$.

~~$(k-1)x = -2$~~

~~$k - 1 = \frac{-2}{x}$~~

~~$k = 1 - \frac{2}{x}$.~~

Let $g(x) = -\frac{2xe^{kx}}{k}$. The diagram below shows sections of the graphs of f and g for $x \geq 0$.



Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

c. Write down a definite integral that gives the value of A. 1 mark

$\int_0^2 f(x)\,dx - \int_0^2 g(x)\,dx$.

$\int_0^2 x^2 e^{kx}\,dx - \int_0^2 -\frac{2xe^{kx}}{k}\,dx = \int_0^2 \left(x^2 e^{kx} + \frac{2xe^{kx}}{k}\right)dx$

d. Using your result from **part a.**, or otherwise, find the value of k such that $A = \frac{16}{k}$. 3 marks

$\int_0^2 \left(x^2 e^{kx} + \frac{2xe^{kx}}{k}\right) dx$.

$\int_0^2 \frac{1}{k} x e^{kx}(kx + 2)\,dx$

$\frac{1}{k}\int_0^2 xe^{kx}(kx+2)\,dx$

$\frac{1}{k}\left[x^2 e^{kx}\right]_0^2$

$\frac{1}{k}\left[2^2 e^{2k} - 0^2 e^0\right] = \frac{4e^{2k}}{k}$.

$A = \frac{16}{k} = \frac{4e^{2k}}{k}$

$4 = e^{2k}$

$\log_e 4 = 2k$

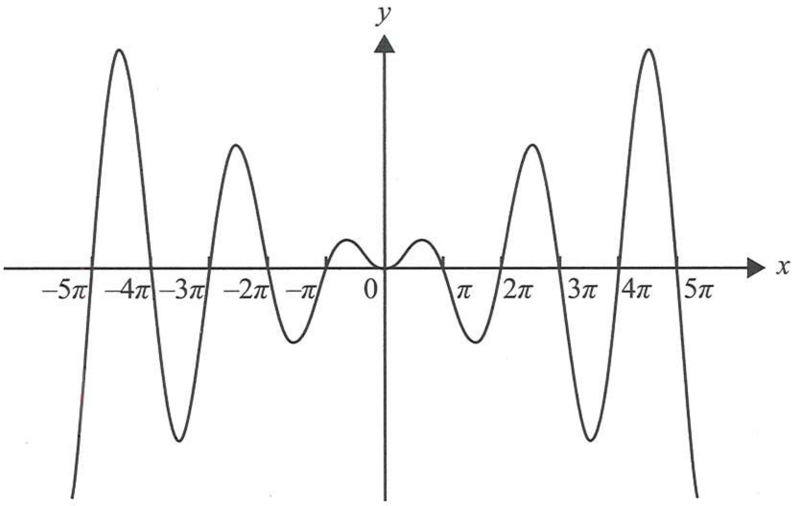
$k = \frac{1}{2}\log_e 4$

$= \log_e 4^{\frac{1}{2}}$

$= \log_e 2$.

Question 9 (8 marks)

Consider a part of the graph of $y = x\sin(x)$, as shown below.



a. **i.** Given that $\int (x\sin(x))dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))dx$ when n is a positive **even** integer or 0. Give your answer in simplest form. 2 marks

n odd. n+1 even.
n → even
n+1 → odd.

$$\int_{n\pi}^{(n+1)\pi} (x\sin x)\,dx = \Big[\sin x - x\cos x\Big]_{n\pi}^{(n+1)\pi}$$

$$= \Big(\sin[(n+1)\pi] - (n+1)\pi\cos[(n+1)\pi]\Big) - \Big(\sin(n\pi) - n\pi\cos(n\pi)\Big)$$

$$= \big(0 - (n+1)\pi \times 1\big) - \big(0 - n\pi \times 1\big)$$

$$= (n+1)\pi + n\pi = (2n+1)\pi$$

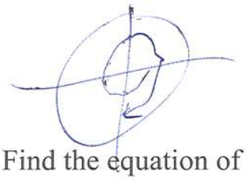
ii. Given that $\int (x\sin(x))dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))dx$ when n is a positive **odd** integer. Give your answer in simplest form. 1 mark

n → odd
n+1 → even

$$\int (x\sin x)\,dx = \big(0 - (n+1)\pi \times 1\big) - \big(0 - n\pi \times -1\big)$$

$$= -(n+1)\pi - n\pi$$

$$= -(2n+1)\pi$$



b. Find the equation of the tangent to $y = x\sin(x)$ at the point $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$. 2 marks

$y = x \sin x.$

$u = x \quad v = \sin x$

$u' = 1 \quad v' = \cos x$

$\frac{dy}{dx} = x\cos(x) + \sin x.$

$M = -\frac{5\pi}{2} \times \cos\left(-\frac{5\pi}{2}\right) + \sin\left(-\frac{5\pi}{2}\right)$

$= -\frac{5\pi}{2} \times 0 + -1 = -1$

$y - \frac{5\pi}{2} = -1\left(x + \frac{5\pi}{2}\right)$

$y - \frac{5\pi}{2} = -x - \frac{5\pi}{2}$

$y = -x.$

c. The translation T maps the graph of $y = x\sin(x)$ onto the graph of $y = (3\pi - x)\sin(x)$, where

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}$$

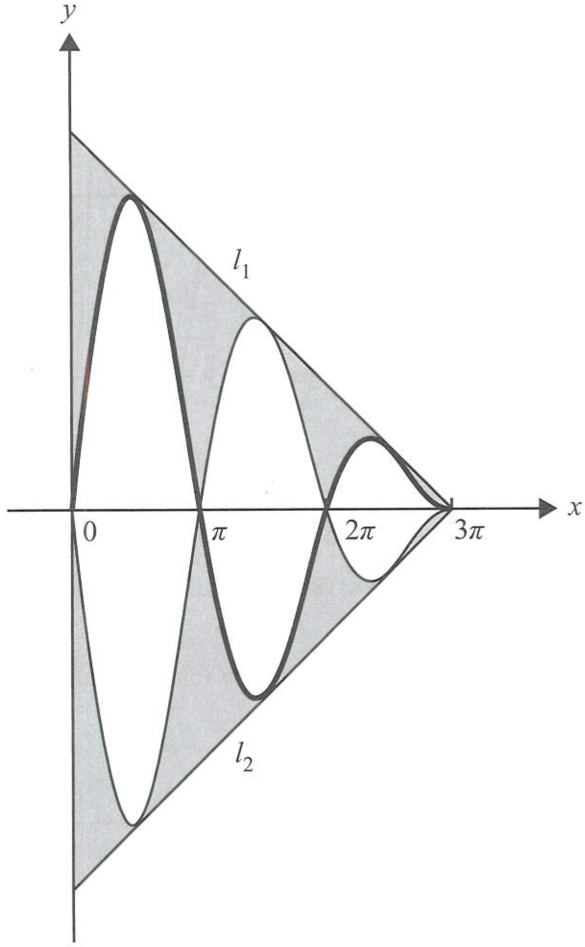
and a is a real constant.

State the value of a. 1 mark

3π

d. Let $f : [0, 3\pi] \rightarrow R,\ f(x) = (3\pi - x)\sin(x)$ and $g : [0, 3\pi] \rightarrow R,\ g(x) = (x - 3\pi)\sin(x)$.

The line l_1 is the tangent to the graph of f at the point $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and the line l_2 is the tangent to the graph of g at $\left(\frac{\pi}{2}, -\frac{5\pi}{2}\right)$, as shown in the diagram below.



Above + Below the x Axis.

Find the total area of the shaded regions shown in the diagram above. 2 marks

$2 \times \left(\int_0^{3\pi} (-x + 3\pi)\,dx - \left[\int_0^{\pi} (3\pi - x)\sin x\,dx - \int_{\pi}^{2\pi} (x - 3\pi)\sin x\,dx + \int_{2\pi}^{3\pi} (3\pi - x)\sin x\,dx \right] \right)$

$2 \times \left(\left[-\frac{x^2}{2} + 3\pi x \right]_0^{3\pi} - \left[\pi - 3\pi + 5\pi \right] \right)$

$2 \times \left(\left(-\frac{9\pi^2}{2} + 9\pi^2 \right) - \left(-\frac{0}{2} + 0 \right) - \left[9\pi \right] \right)$

$2 \times \left(\frac{9\pi^2}{2} - 9\pi \right)$

$9\pi^2 - 18\pi = 9\pi(\pi - 2)$

END OF QUESTION AND ANSWER BOOK