VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2019

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

ANS

STUDENT NUMBER | | | | | | | | | | Letter

MATHEMATICAL METHODS

Written examination 1

Wednesday 6 November 2019

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	9	40

- Students are to write in blue or black pen.
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 13 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (4 marks)

Let $f: \left(\frac{1}{3}, \infty\right) \to R,\ f(x) = \frac{1}{3x-1}$.

a. **i.** Find $f'(x)$. 1 mark

$f(x) = (3x-1)^{-1}$

$f'(x) = -1\,(3x-1)^{-2} \times 3$

$= \frac{-3}{(3x-1)^2}\ ,\ x > \frac{1}{3}.$

ii. Find an antiderivative of $f(x)$. 1 mark

$\int \frac{1}{3x-1}\,dx = \frac{1}{3}\log_e(3x-1)$

b. Let $g: R\setminus\{-1\} \to R,\ g(x) = \frac{\sin(\pi x)}{x+1}$.

$u = \sin(\pi x)$ $v = (x+1)^{-1}$

$u' = \pi\cos(\pi x)$ $v' = -1\,(x+1)^{-2}.$

Evaluate $g'(1)$. 2 marks

$g(x) = \sin(\pi x) \times (x+1)^{-1}$

$g'(x) = \sin(\pi x) \times -(x+1)^{-2} + (x+1)^{-1}\,\pi\cos(\pi x).$

$g'(1) = \sin(\pi) \times -(1+1)^{-2} + (1+1)^{-1} \times \pi\cos(\pi).$

$= 0 + \frac{\pi \times -1}{2}$

$=$

$= -\frac{\pi}{2}.$

DO NOT WRITE IN THIS AREA

TURN OVER

Question 2 (4 marks)

a. Let $f: R \setminus \left\{\frac{1}{3}\right\} \to R, f(x) = \frac{1}{3x-1}$.

Find the rule of f^{-1}. 2 marks

$x = \frac{1}{3y-1}$

$3y-1 = \frac{1}{x}$ $\longrightarrow$ $y - \frac{1}{3} = \frac{1}{3x}$

$3y = \frac{1}{x} + 1$

$y = \frac{1}{3x} + \frac{1}{3}$

$f^{-1}(x) = \frac{1}{3x} + \frac{1}{3} = \frac{1+x}{3x}$

b. State the domain of f^{-1}. 1 mark

$R \setminus \{0\}$

c. Let g be the function obtained by applying the transformation T to the function f, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

and $c, d \in R$.

Find the values of c and d given that $g = f^{-1}$. 1 mark

$x' = x + c$ $\quad$ $x = x' - c$

$y' = y + d$ $\quad$ $y = y' - d$

$y' - d = \frac{1}{3(x'-c)-1}$

$y' - d = \frac{1}{3x' - 3c - 1}$

$-3c - 1 = 0$

$-3c = 1$

$c = -\frac{1}{3}$

$c = -\frac{1}{3}$

$d = +\frac{1}{3}$

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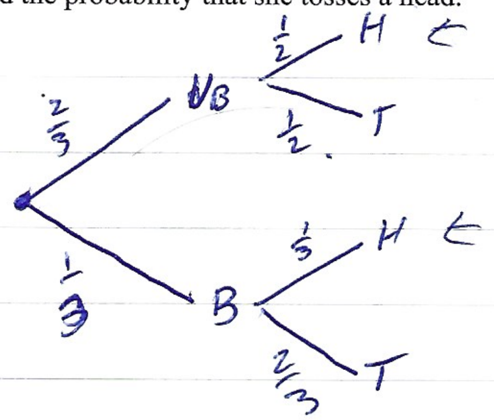
Question 3 (3 marks)

The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail.

Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$.

Jo randomly selects a coin from her pocket and tosses it.

a. Find the probability that she tosses a head. 2 marks



$\frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3}$

$\frac{1}{3} + \frac{1}{9}$

$\frac{3}{9} + \frac{1}{9}$

$\frac{4}{9}$

b. Find the probability that she selected an unbiased coin, given that she tossed a head. 1 mark

$Pr(U|H) = \frac{Pr(U \cap H)}{Pr(H)}$

$= \frac{1/3}{4/9}$

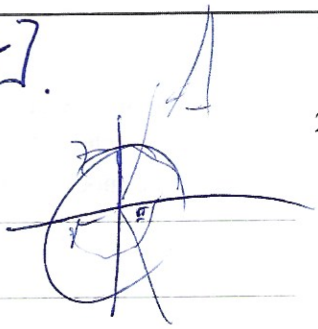
$= \frac{1}{\cancel{3}} \times \frac{\cancel{9}\,3}{4}$

$= \frac{3}{4}$

TURN OVER

Question 4 (4 marks)

$\frac{x}{2} \in [-\pi, \pi]$.



a. Solve $1-\cos\left(\frac{x}{2}\right)=\cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$. 2 marks

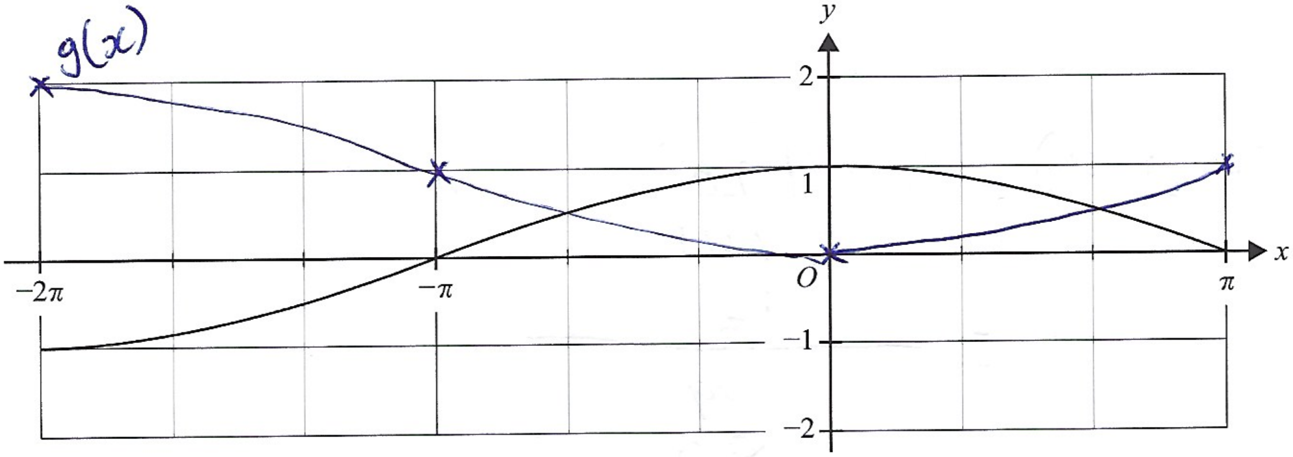
$1 = 2\cos\left(\frac{x}{2}\right)$

$\frac{1}{2} = \cos\left(\frac{x}{2}\right)$.

$\frac{x}{2} = -\frac{\pi}{3}, \frac{\pi}{3}$.

$x = -\frac{2\pi}{3}, \frac{2\pi}{3}$.

b. The function $f: [-2\pi, \pi] \to R$, $f(x) = \cos\left(\frac{x}{2}\right)$ is shown on the axes below.



Let $g: [-2\pi, \pi] \to R$, $g(x) = 1 - f(x)$.

Sketch the graph of g on the axes above. Label all points of intersection of the graphs of f and g, and the endpoints of g, with their coordinates. 2 marks

Question 5 (5 marks)

Let $f: R\setminus\{1\} \to R,\ f(x) = \dfrac{2}{(x-1)^2} + 1$.

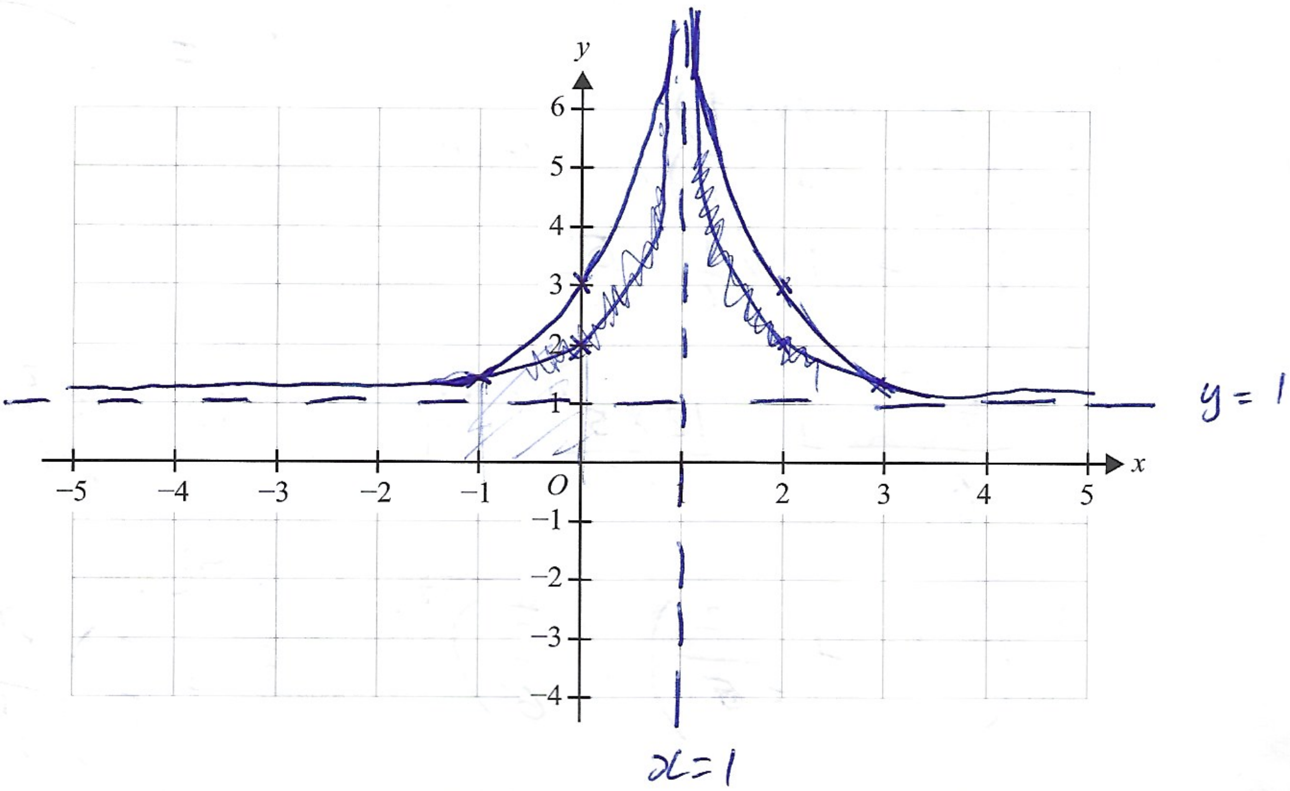
a. **i.** Evaluate $f(-1)$. 1 mark

$f(-1) = \dfrac{2}{(-1-1)^2} + 1$

$= \dfrac{2}{4} + 1$

$= 3/2.$

ii. Sketch the graph of f on the axes below, labelling all asymptotes with their equations. 2 marks



b. Find the area bounded by the graph of f, the x-axis, the line $x = -1$ and the line $x = 0$. 2 marks

$\int_{-1}^{0} \dfrac{2}{(x-1)^2} + 1 \, dx.$

$\int 2(x-1)^{-2} + 1 \, dx$

$\dfrac{2(x-1)^{-1}}{-1} + x.$

$= \left[\dfrac{-2}{x-1} + x \right]_{-1}^{0}$

$= \left[\dfrac{-2}{0-1} + 0 \right] - \left[\dfrac{-2}{\cancel{-1-1}\,-2} - 1 \right]$

$= (2) - (1-1)$

$= 2.$

Question 6 (3 marks)

Fred owns a company that produces thousands of pegs each day. He randomly selects 41 pegs that are produced on one day and finds eight faulty pegs.

a. What is the proportion of faulty pegs in this sample? 1 mark

$\frac{8}{41}$

b. Pegs are packed each day in boxes. Each box holds 12 pegs. Let $\hat{P}$ be the random variable that represents the proportion of faulty pegs in a box.

The actual proportion of faulty pegs produced by the company each day is $\frac{1}{6}$.

$Pr(X < 2).$

Find $\Pr\left(\hat{P} < \frac{1}{6}\right)$. Express your answer in the form $a(b)^n$, where a and b are positive rational numbers and n is a positive integer. 2 marks

$12 \times \frac{1}{6} = 2.$

$Pr(x=0) + Pr(x=1)$

$\binom{12}{0}\left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^{12} + \binom{12}{1} \times \frac{1}{6} \times \left(\frac{5}{6}\right)^{11}$

$\frac{5^{12}}{6^{12}} + 12 \times \frac{1}{6} \times \frac{5^{11}}{6^{11}}$

$\frac{5^{12}}{6^{12}} + \frac{12 \times 5^{11}}{6^{12}} \rightarrow \frac{12 \times 5^{12}}{5 \; 6^{12}}$

$\left(1 + \frac{12}{5}\right)\left(\frac{5}{6}\right)^{12}$

$\left(\frac{5+12}{5}\right)\left(\frac{5}{6}\right)^{12}$

$\frac{17}{5}\left(\frac{5}{6}\right)^{12}$

OR.

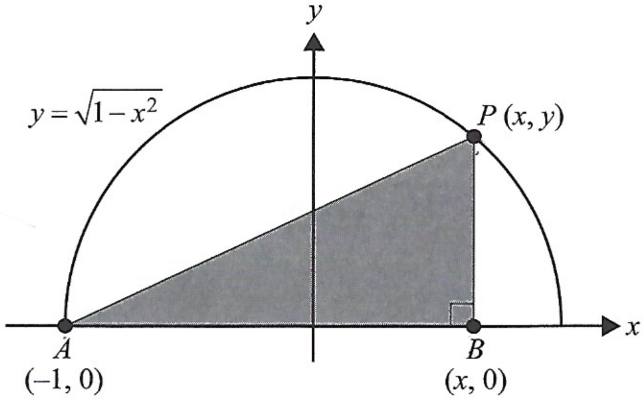
$\left(\frac{5}{6} + \frac{12}{6}\right)\frac{5^{11}}{6^{11}}.$

$\frac{17}{6}\left(\frac{5}{6}\right)^{11}$

DO NOT WRITE IN THIS AREA

Question 7 (4 marks)

The graph of the relation $y = \sqrt{1 - x^2}$ is shown on the axes below. P is a point on the graph of this relation, A is the point (–1, 0) and B is the point (x, 0).



a. Find an expression for the length PB in terms of x only. 1 mark

$\sqrt{1 - x^2}$

b. Find the maximum area of the triangle ABP. 3 marks

$A = \frac{1}{2} \times (1 + x) \times \sqrt{1 - x^2}$

$A = \frac{1}{2}(x + 1)(1 - x^2)^{\frac{1}{2}}$.

$u = \frac{1}{2}(x + 1) \quad u' = \frac{1}{2}$

$v = (1 - x^2)^{\frac{1}{2}}$

$v' = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \times -2x$

$= -\frac{x}{\sqrt{1 - x^2}}$

$A' = \frac{1}{2}(x + 1) \times \frac{-x}{\sqrt{1 - x^2}} + (1 - x^2)^{\frac{1}{2}} \times \frac{1}{2}$

$A' = \frac{-x(x + 1)}{2\sqrt{1 - x^2}} + \frac{\sqrt{1 - x^2}}{2}$.

$A' = \frac{-x(x + 1) + (1 - x^2)}{2\sqrt{1 - x^2}}$

$A' = \frac{-x^2 - x + 1 - x^2}{2\sqrt{1 - x^2}}$.

$A' = \frac{1 - x - 2x^2}{2\sqrt{1 - x^2}}$

$A' = 0$.

$1 - x - 2x^2 = 0$

$(1 - 2x)(1 + x) = 0$

$1 + x = 0 \quad 1 - 2x = 0$

$x = -1 \quad 1 = 2x$

$x = \frac{1}{2}$.

$1 - \frac{1}{4}$

$\frac{3}{4}$

$A = \frac{1}{2}\left(\frac{1}{2} + 1\right) \times \sqrt{1 - \left(\frac{1}{2}\right)^2}$

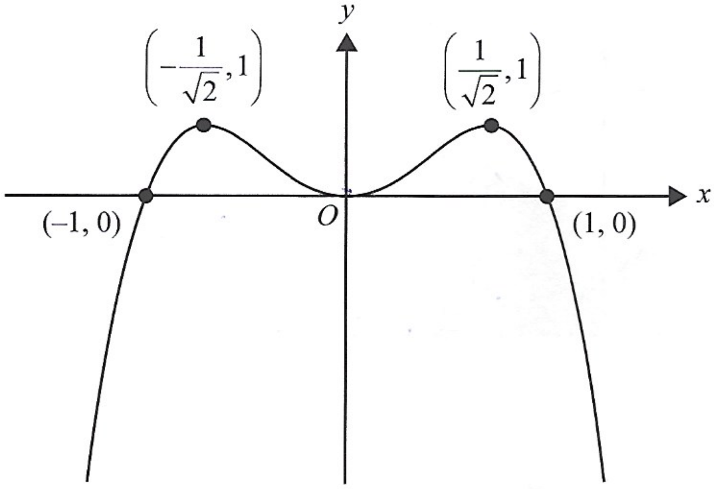
$= \frac{1}{2}\left(\frac{3}{2}\right) \times \sqrt{1 - \frac{1}{4}}$

$= \frac{1}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{8}$ sq units

Question 8 (4 marks)

The function $f: R \to R, f(x)$ is a polynomial function of degree 4. Part of the graph of f is shown below. The graph of f touches the x-axis at the origin.



a. Find the rule of f. 1 mark

$f(x) = k\,x^2(x+1)(x-1)$

$f(x) = k\,x^2(x^2-1)$

$f\left(\frac{1}{\sqrt{2}}\right) = 1 = k \times \left(\frac{1}{\sqrt{2}}\right)^2\left(\left(\frac{1}{\sqrt{2}}\right)^2 - 1\right)$

$1 = k \times \frac{1}{2} \times \left(\frac{1}{2} - 1\right)$

$1 = k \times \frac{1}{2} \times -\frac{1}{2}$

$1 = k \times -\frac{1}{4}$

$k = -4$

$f(x) = -4x^2(x^2-1)$.

Let g be a function with the same rule as f.

Let $h: D \to R$, $h(x) = \log_e(g(x)) - \log_e(x^3 + x^2)$, where D is the maximal domain of h.

b. State D. 1 mark

$g(x). \to (-1, 0) \cup (0, 1)$. $x^3 + x^2. \to (-1,0) \cup (0, \infty)$.

$D_h: (-1, 0) \cup (0, 1). \to$ OR $(-1, 1) \setminus \{0\}$

$h(x) = \log_e\left(\frac{-4\cancel{x^2}(x-1)\cancel{(x+1)}}{\cancel{x^2}\cancel{(x+1)}}\right)$

$= \log_e(-4(x-1))$

$= \log_e(4 - 4x)$

Question 8 – continued

1.386.

c. State the range of h. 2 marks

~~$(0, \log_e 4)$~~

$x = -1$ $h \to \log_e (4+4)$

$h \to \log_e 8$.

$x = 1$ $h \to \log_e (4-4)$

~~$(\log_e 4)$~~

$h \to \log_e (0)$.

$\to -\infty$

~~$(-\infty, \log_e 4)$~~. $x = 0$. $h \to \log_e (4)$.

~~$(-\infty, \log_e 4) \cup (\log_e 4, 2)$~~

$(-\infty, \log_e 4) \cup (\log_e 4, \log_e 8)$

OR $(-\infty, \log_e 8) \setminus \log_e 4$.

Question 9 (9 marks)

Consider the functions $f: R \to R, f(x) = 3 + 2x - x^2$ and $g: R \to R, g(x) = e^x$.

a. State the rule of $g(f(x))$. 1 mark

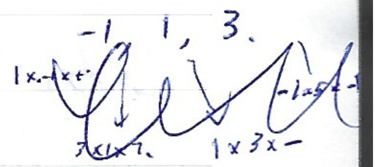
$g(f(x)) = e^{3+2x-x^2}$

b. Find the values of x for which the derivative of $g(f(x))$ is negative. 2 marks

$g'(f(x)) = e^{3+2x-x^2} \times (2-2x).$

$= (2-2x)\, e^{3+2x-x^2}$

$= 2(1-x)\, e^{3+2x-x^2}$



$1 - x < 0$

$x > 1$

c. State the rule of $f(g(x))$. 1 mark

$f(g(x)) = 3 + 2e^x - (e^x)^2 = 3 + 2e^x - e^{2x}.$

d. Solve $f(g(x)) = 0$. 2 marks

$f(g(x)) = 0 = 3 + 2e^x - e^{2x}.$

Let $A = e^x$

$3 + 2A - A^2$

$(3 - A)(1 + A)$

$0 = (3 - e^x)(1 + e^x)$

$3 - e^x = 0$ $\quad$ $1 + e^x = 0$

$e^x = 3.$ $\quad$ $e^x = -1$

$x = \log_e(3).$ $\quad$ No Sol^n.

Question 9 – continued

e. Find the coordinates of the stationary point of the graph of $f(g(x))$. 2 marks

$f'(g(x)) = 2e^{x} - 2e^{2x} = 0$

$0 = 2e^{x}(1 - e^{x})$

$2e^{x} = 0$ No Sol^n $\quad 1 - e^{x} = 0$

$e^{x} = 1$

$x = \log_e(1)$.

$x = 0$.

~~$f(g(x)) = 3+2e - e^{2}$~~

$f(g(0)) = 3 + 2 \times e^{0} - e^{0}$

$= 3 + 2 - 1$

$= 4$

~~$(0, \log(3+2e-e))$~~.

$(0, 4)$.

f. State the number of solutions to $g(f(x)) + f(g(x)) = 0$. 1 mark

One.

$3 + 2e^{x} - e^{2x} + e^{3+2x-x^2} = 0$.

$3 + 2e^{x} - e^{2x} + \dfrac{e^{3} \times e^{2x}}{e^{x^2}} = 0$

$+ e^{3} \times e^{x} \times e^{x}$

$f(g(x)) = 3 + 2e^{x} - e^{2x}$

from Part d has one x int at $x = \log_e(3)$

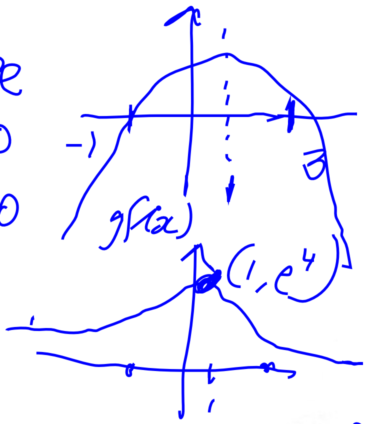
as $x \to -\infty$ $f(g(x)) \to 3$

when $x = 0$ $f(g(x)) = 3 + 2e^{0} - e^{0}$
$= 4$

as $x \to \infty$ $f(g(x)) \to 3 + 2e^{\infty} - (e^{\infty})^{2}$
$\to \infty - \infty^{2}$
$\to -\infty$

$g(f(x)) = e^{3+2x-x^2}$
$= e^{(3-x)(1+x)}$

$3 + 2x - x^{2}$ looks like

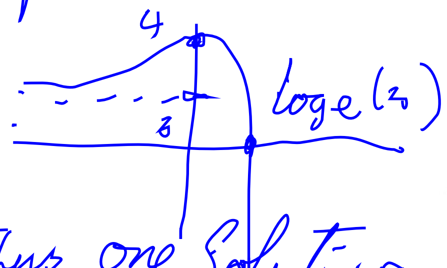


as $x \to -\infty$ $g(f(x)) \to 0$

as $x \to \infty$ $g(f(x)) \to 0$

when $x = 1$ $g(f(x)) = e^{4}$

Graph looks like



Thus one Solution when add ordinates of the graphs.

END OF QUESTION AND ANSWER BOOK