VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2020

ANS

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | Letter

MATHEMATICAL METHODS

Written examination 1

Tuesday 17 November 2020

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 14 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (3 marks)

a. Let $y = x^2 \sin(x)$.

$u = x^2 \quad v = \sin x$

$u' = 2x \quad v' = \cos x.$

Find $\frac{dy}{dx}$. 1 mark

$x^2 \cos(x) + 2x \sin(x).$

b. Evaluate $f'(1)$, where $f : R \to R$, $f(x) = e^{x^2 - x + 3}$. 2 marks

$f'(x) = e^{x^2-x+3} \times (2x-1)$

$= (2x-1)e^{x^2-x+3}$

$f'(1) = (2-1)e^{1-1+3}$

$= 1 \times e^3$

$= e^3$

Question 2 (3 marks)

A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

a. State the probability that at any given six-month service model X will require an air filter change without an oil change. 1 mark

$\frac{3}{20} - \frac{1}{20} = \frac{2}{20} = \frac{1}{10}$.

b. The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^+$. ← Integers, Positive

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05 2 marks

$\frac{n}{m+n} - \frac{1}{m+n} = \frac{n-1}{m+n} = 0.05$

$n - 1 = 0.05m + 0.05n$.

$0.95n - 1 = 0.05m$.

$m = \frac{0.95n - 1}{0.05}$

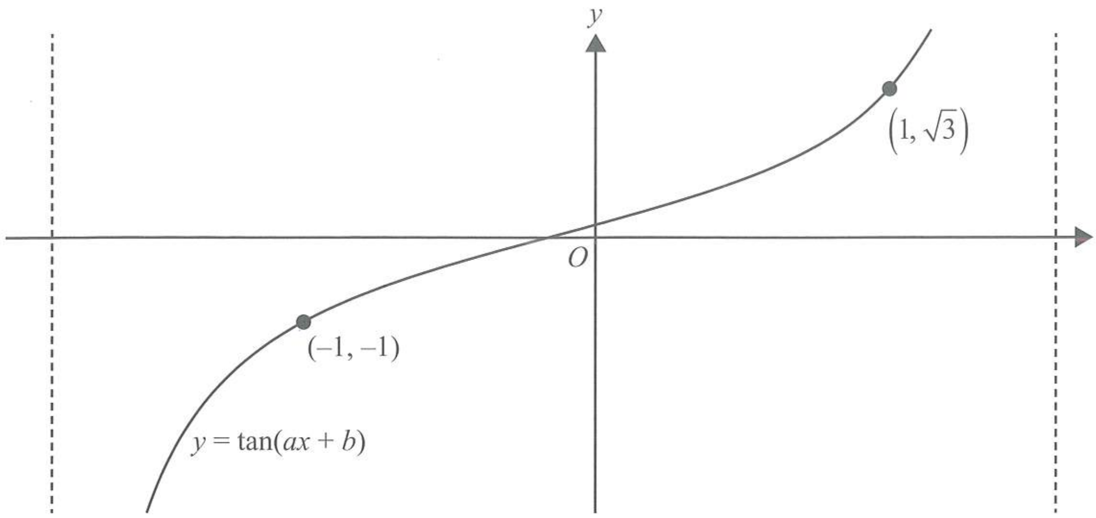
$m = \frac{0.95}{0.05}n - \frac{1}{0.05}$

$m = 19n - 20$.

DO NOT WRITE IN THIS AREA

Question 3 (3 marks)

Shown below is part of the graph of a period of the function of the form $y = \tan(ax + b)$.



The graph is continuous for $x \in [-1, 1]$.

Find the value of a and the value of b, where $a > 0$ and $0 < b < 1$.

$-1 = \tan(\cancel{a} + b)$ $\sqrt{3} = \tan(a+b)$.

$-1 = \tan(-a+b)$ $\sqrt{3} = \tan(a+b)$.

$-a+b = -\frac{\pi}{4}$ $a+b = \frac{\pi}{3}$.

$2b = \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{12}$

$b = \frac{\pi}{24}$

$a + \frac{\pi}{24} = \frac{\pi}{3}$

$a = \frac{\pi}{3} - \frac{\pi}{24}$

$= \frac{8\pi}{24} - \frac{\pi}{24}$

$= \frac{7\pi}{24}$.

Question 4 (3 marks)

Solve the equation $2\log_2(x+5) - \log_2(x+9) = 1$.

$\log_2 (x+5)^2 - \log_2 (x+9) = 1$

$\log_2 \left(\frac{(x+5)^2}{(x+9)} \right) = 1$

$\frac{(x+5)^2}{x+9} = 2^1$

$(x+5)^2 = 2(x+9)$

$x^2 + 10x + 25 = 2x + 18$

$x^2 + 8x + 7 = 0$

$(x+1)(x+7) = 0$

$x = -1 \quad x = -7$

Not Possible as $x+5$ would be -ve

$x = -1$.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

0 1 2 3 4

2

3

7

257
× 8
216

685
− 16
609

56
× 24
228

DO NOT WRITE IN THIS AREA

Question 5 (4 marks)

For a certain population the probability of a person being born with the specific gene SPGE1 is $\frac{3}{5}$.

The probability of a person having this gene is independent of any other person in the population having this gene.

Binomial

a. In a randomly selected group of four people, what is the probability that three or more people have the SPGE1 gene? 2 marks

$Pr(X=3)+Pr(X=4)=\binom{4}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^1+\binom{4}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^0$

$=4\times\left(\frac{27}{125}\right)\left(\frac{2}{5}\right)+1\times\frac{81}{625}$

$=\frac{216}{625}+\frac{81}{625}$

$=\frac{297}{625}.$

b. In a randomly selected group of four people, what is the probability that exactly two people have the SPGE1 gene, given that at least one of those people has the SPGE1 gene? Express your answer in the form $\frac{a^3}{b^4-c^4}$, where $a, b, c \in Z^+$. 2 marks

$2\times3\times3^2\times2^2$

$3^3\times2^3$

6^3

$Pr(X=2)=\binom{4}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2=6\times\frac{9}{25}\times\frac{4}{25}=\frac{224}{625}$

$Pr(X\geq1)=1-Pr(X=0)=1-\binom{4}{0}\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^4$

$=1-1\times1\times\frac{16}{625}$

$=\frac{609}{625}$

$Pr(X=2|X\geq1)=\cancel{\frac{\frac{224}{625}}{\frac{609}{625}}}$

$=\frac{224}{609}$

$=\frac{6^3}{5^4-2^4}$

24-07 7006970

Question 6 (8 marks)

Let $f: [0, 2] \rightarrow R$, where $f(x) = \frac{1}{\sqrt{2}}\sqrt{x}$.

a. Find the domain and the rule for f^{-1}, the inverse function of f. 2 marks

$x = \frac{1}{\sqrt{2}}\sqrt{y}$ $f^{-1}(x) = 2x^2$

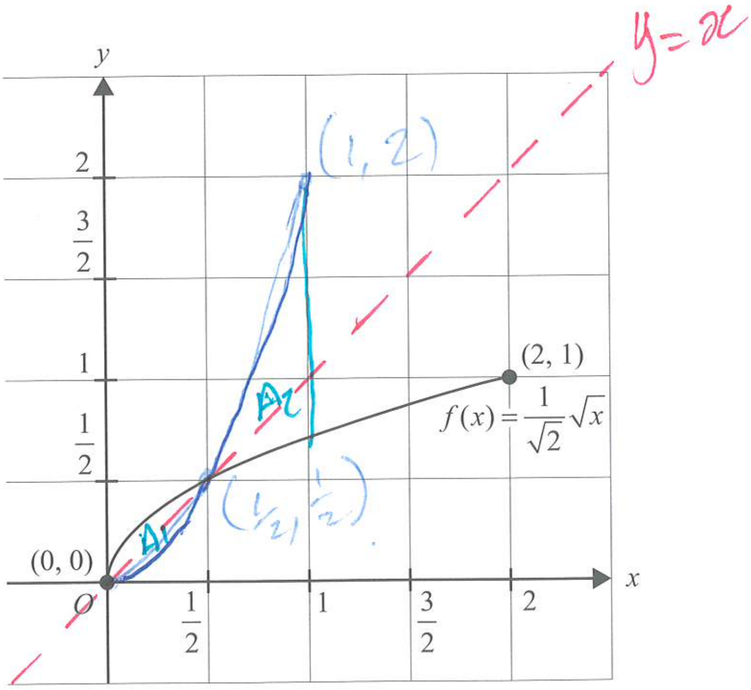
$\sqrt{2}\,x = \sqrt{y}$.

$2x^2 = y$ Domain f^{-1} $[0, 1]$

$f(0) = \frac{1}{\sqrt{2}} \times \sqrt{0} = 0$

$f(2) = \frac{1}{\sqrt{2}} \times \sqrt{2} = 1$

The graph of $y = f(x)$, where $x \in [0, 2]$, is shown on the axes below.



b. On the axes above, sketch the graph of f^{-1} over its domain. Label the endpoints and point(s) of intersection with the function f, giving their coordinates. 2 marks

c. Find the total area of the two regions: one region bounded by the functions f and f^{-1}, and the other region bounded by f, f^{-1} and the line $x = 1$. Give your answer in the form $\frac{a - b\sqrt{b}}{6}$, where $a, b \in Z^{+}$. 4 marks

A1 A2

$A_1 = \int_0^{\frac{1}{2}} \left[\frac{1}{\sqrt{2}}\sqrt{x} - 2x^2\right] dx$ $x^{+\frac{1}{2}}$

$= \left[\frac{2x^{\frac{3}{2}}}{3\sqrt{2}} - \frac{2x^3}{3}\right]_0^{\frac{1}{2}}$

$= \left(\frac{2(\frac{1}{2})^{\frac{3}{2}}}{3\sqrt{2}} - \frac{2(\frac{1}{2})^3}{3}\right) - (0)$

$= \left(\frac{\cancel{2}}{3\sqrt{2}} \times \frac{1}{\cancel{8}\sqrt{2}} - \frac{\cancel{2}}{3} \times \frac{1}{\cancel{8}4}\right)$

$= \frac{1}{6} - \frac{1}{12}$

$= \frac{2}{12} - \frac{1}{12}$

$= \frac{1}{12}$

$A_2 = \int_{\frac{1}{2}}^{1} \left[2x^2 - \frac{1}{\sqrt{2}}\sqrt{x}\right] dx$

$= \left[\frac{2x^3}{3} - \frac{2x^{\frac{3}{2}}}{3\sqrt{2}}\right]_{\frac{1}{2}}^{1}$

$= \left(\frac{2 \times 1^3}{3} - \frac{2 \times 1^{\frac{3}{2}}}{3\sqrt{2}}\right) - \left(\frac{2(\frac{1}{2})^3}{3} - \frac{2(\frac{1}{2})^{\frac{3}{2}}}{3\sqrt{2}}\right)$

$= \left(\frac{2}{3} - \frac{2}{3\sqrt{2}}\right) - \left(\frac{1}{12} - \frac{1}{6}\right)$

$= \left(\frac{2}{3} - \frac{2\sqrt{2}}{6}\right) - \left(-\frac{1}{12}\right)$

$= \left(\frac{4 - 2\sqrt{2}}{6}\right) + \frac{1}{12}$

$= \left(\frac{8 - \cancel{4}\sqrt{2}}{12}\right) + \frac{1}{12}$

$= \frac{9 - 2\sqrt{2}}{12}$

$\text{Area} = \frac{1}{12} + \frac{9 - \cancel{4}\sqrt{2}}{12}$

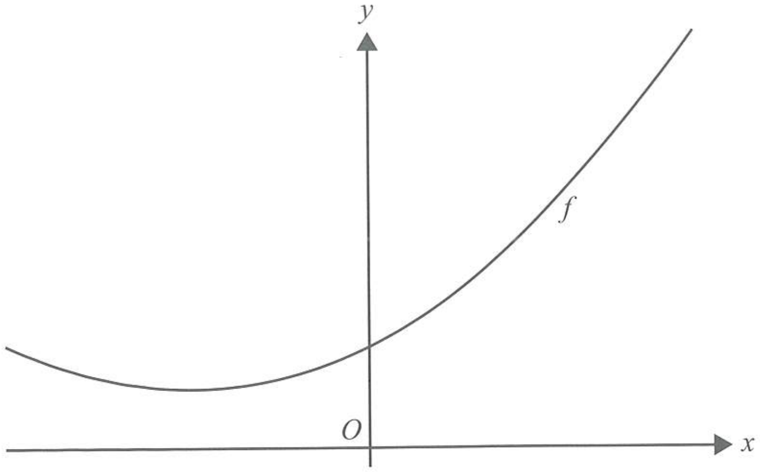
$= \frac{10 - \cancel{4}\sqrt{2}}{12}$

$= \frac{\cancel{2}(5 - 2\sqrt{2})}{\cancel{12}6}$

$= \frac{5 - 2\sqrt{2}}{6}$

Question 7 (8 marks)

Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.



a. Show that point P is not on the graph of $y = f(x)$. 1 mark

$f(1) = 1^2 + 3 \times 1 + 5$ Point on graph $(1, 9)$.

$= 1 + 3 + 5$ $\therefore$ $P(1, 0)$ not on graph.

$= 9$.

b. Consider a point $Q(a, f(a))$ to be a point on the graph of f.

i. Find the slope of the line connecting points P and Q in terms of a. 1 mark

$M_{PQ} = \dfrac{f(a) - 0}{a - 1} = \dfrac{a^2 + 3a + 5}{a - 1}$

ii. Find the slope of the tangent to the graph of f at point Q in terms of a. 1 mark

$M_T = f'(x)$ $f'(x) = 2x + 3$.

$f'(a) = 2a + 3$.

M_T at Point Q $= 2a + 3$

Question 7 – continued

DO NOT WRITE IN THIS AREA

iii. Let the tangent to the graph of f at $x = a$ pass through point P.

Find the values of a. 2 marks

$\frac{a^2+3a+5}{a-1} = 2a+3$

$a^2+3a+5 = (2a+3)(a-1)$

$a^2+3a+5 = 2a^2+a-3$

$0 = a^2-2a-8.$

$0 = (a-4)(a+2)$

$a = -2, 4$

iv. Give the equation of one of the lines passing through point P that is tangent to the graph of f. 1 mark

$a=4$ $m_T = 2\times4+3 = 8+3 = 11$ $P(1,0)$

$y-0 = 11(x-1)$

$y = 11x-11$

$a=-2$

$m_T = 2\times-2+3 = -1$

$y-0 = -1(x-1)$

$y = -x+1$

c. Find the value, k, that gives the shortest possible distance between the graph of the function of $y = f(x-k)$ and point P. 2 marks

for $y=f(x)$.

$m_T = 0 \rightarrow 2x+3=0$ T.P. above $P(1,0)$.

$x = -\frac{3}{2}.$

Need to translate to the right by $\frac{5}{2}$ units

$k = \frac{5}{2}.$

Question 8 (8 marks)

Part of the graph of $y = f(x)$, where $f: (0, \infty) \to R, f(x) = x \log_e(x)$, is shown below.

x int

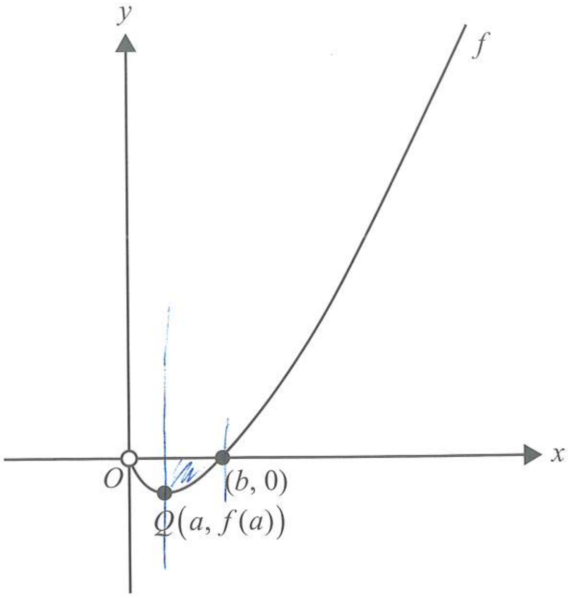
$0 = x \log_e(x)$.

$x = 0 \quad \log_e(x) = 0$

$x = e^0$

$= 1$

$\Rightarrow b = 1$



The graph of f has a minimum at the point $Q(a, f(a))$, as shown above.

a. Find the coordinates of the point Q. 2 marks

$f'(x) = 0$

$u = x \quad v = \log_e(x)$

$u' = 1 \quad v' = \frac{1}{x}$

$f'(x) = x \times \frac{1}{x} + 1 \times \log_e(x)$

$f'(x) = 1 + \log_e(x)$

$0 = 1 + \log_e(x)$.

$\log_e(x) = -1$

$x = e^{-1} = \frac{1}{e}$.

$f(\frac{1}{e}) = \frac{1}{e} \log_e(\frac{1}{e})$.

$= \frac{1}{e} \times -1$

$= -\frac{1}{e}$

$Q(\frac{1}{e}, -\frac{1}{e})$

DO NOT WRITE IN THIS AREA

Question 8 – continued

b. Using $\frac{d\left(x^2\log_e(x)\right)}{dx} = 2x\log_e(x) + x$, show that $x\log_e(x)$ has an antiderivative $\frac{x^2\log_e(x)}{2} - \frac{x^2}{4}$. 1 mark

$x\log_e(x) = \frac{1}{2}\frac{d}{dx}\left(x^2\log_e(x)\right) - \frac{x}{2}.$

$\int x\log_e(x) = \frac{1}{2}\int\left[\frac{d}{dx}\left(x^2\log_e(x)\right) - x\right]dx$

$= \frac{1}{2}\left[x^2\log_e(x) - \frac{x^2}{2}\right] + c.$

$= \frac{x^2\log_e(x)}{2} - \frac{x^2}{4} + c$

when $c = 0$ $= \frac{x^2\log_e(x)}{2} - \frac{x^2}{4}.$

c. Find the area of the region that is bounded by f, the line $x = a$ and the horizontal axis for $x \in [a, b]$, where b is the x-intercept of f. 2 marks

Area $= -\int_a^b x\log_e(x)\,dx.$

$= -\left[\frac{x^2\log_e(x)}{2} - \frac{x^2}{4}\right]_a^b.$

$= -\left[\left(\frac{b^2\log_e(b)}{2} - \frac{b^2}{4}\right) - \left(\frac{a^2\log_e(a)}{2} - \frac{a^2}{4}\right)\right]$

when $a = +\frac{1}{e}$ $b = 1$

Area $= -\left[\left(\frac{1^2\log_e(1)}{2} - \frac{1^2}{4}\right) - \left(\frac{(\frac{1}{e})^2\log_e(+\frac{1}{e})}{2} - \frac{(\frac{1}{e})^2}{4}\right)\right]$

$= -\left[\left(0 - \frac{1}{4}\right) - \left(\frac{1}{2e^2} \times -1 - \frac{1}{4e^2}\right)\right]$

$= -\left[-\frac{1}{4} + \frac{2}{4e^2} + \frac{1}{4e^2}\right]$

$= -\left[-\frac{1}{4} + \frac{3}{4e^2}\right]$

$= \frac{1e^2}{4e^2} - \frac{3}{4e^2} = \frac{e^2 - 3}{4e^2}$

d. Let $g : (a, \infty) \to R$, $g(x) = f(x) + k$ for $k \in R$.

$g'(x) = 2.$

i. Find the value of k for which $y = 2x$ is a tangent to the graph of g. 1 mark

$y = 2e.$ $(e, 2e).$

$g'(x) = 1 + \log_e(x).$

$2 = 1 + \log_e(x)$

$1 = \log_e(x).$

$x = e$

$g(e) = 2e = e\log_e(e) + k.$

$2e = e + k$

$k = e$

ii. Find all values of k for which the graphs of g and g^{-1} do not intersect. 2 marks

g and g^{-1} intersect on $y = x.$

gradient is 1 it Touch as $y = x$ is tangent to both.

$g(x) = x\log_e(x) + k.$

$x = x\log_e(x) + k.$

$x - x\log_e(x) = k.$

~~Min when~~ $x = \frac{1}{e}$

$\cancel{\frac{1}{e} - \frac{1}{e}\log_e(e^{-1}) = k.}$

$\cancel{\frac{1}{e} + \frac{1}{e} = k.}$

$\cancel{\frac{2}{e} = k.}$

$\cancel{k > \frac{2}{e}.}$

$g'(x) = 1$

$1 = 1 + \log_e(x)$

$0 = \log_e(x)$

$x = e^0 = 1.$

$1 - 1\log_e(1) = k.$

$1 - 0 = k$

$k = 1$

$\therefore k > 1$

$g(x)$ and $g^{-1}(x)$ won't Intersect.

END OF QUESTION AND ANSWER BOOK