VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2020

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | | Letter | |

SPECIALIST MATHEMATICS

Written examination 1

Thursday 19 November 2020

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 10 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

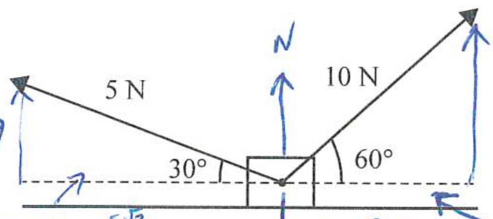
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (5 marks)

Use Exact Values. as Q^n doesn't ask for Decimal.

A 2 kg mass is initially at rest on a smooth horizontal surface. The mass is then acted on by two constant forces that cause the mass to move horizontally. One force has magnitude 10 N and acts in a direction 60° upwards from the horizontal, and the other force has magnitude 5 N and acts in a direction 30° upwards from the horizontal, as shown in the diagram below.

$= 5\sin 30$
$= 5 \times \frac{1}{2}$
$= \frac{5}{2}$



$\leftarrow 10\sin 60 = 10 \times \frac{\sqrt{3}}{2}$
$= 5\sqrt{3}$ N

$5\cos 30 = \frac{5\sqrt{3}}{2}$ $w = 2g$ $10\cos 60 = 5$

a. Find the normal reaction force, in newtons, that the surface exerts on the mass. 2 marks

$N + \frac{5}{2} + 5\sqrt{3} = 2g.$

$N = 2g - \frac{5}{2} - 5\sqrt{3}.$ N.

b. Find the acceleration of the mass, in ms^{-2}, after it begins to move. 2 marks

$R = ma.$ $a = \frac{10 - 5\sqrt{3}}{4}$ m/s^2 Right.

$5 - \frac{5\sqrt{3}}{2} = 2a.$

$\frac{10 - 5\sqrt{3}}{2} = 2a$

c. Find how far the mass travels, in metres, during the first four seconds of motion. 1 mark

$u = 0$ $t = 4$ $a = \frac{10 - 5\sqrt{3}}{4}$ x.

$x = ut + \frac{1}{2}at^2$

$= 0 \times 4 + \frac{1}{2} \times \frac{10 - 5\sqrt{3}}{4} \times 4^2.$

$= \frac{\cancel{16}^{2}(10 - 5\sqrt{3})}{\cancel{8}\,1} = 20 - 10\sqrt{3}$ m.

Question 2 (4 marks)

Evaluate $\int_{-1}^{0} \frac{1+x}{\sqrt{1-x}}\,dx$. Give your answer in the form $a\sqrt{b}+c$, where $a, b, c \in R$.

$2^{\frac{3}{2}} = (2^3)^{\frac{1}{2}} = 8^{\frac{1}{2}} = 2\sqrt{2}$

Let $u = 1-x$. $x=-1$ $u=1--1=2$. $x=0$ $u=1-0=1$.

$x = 1-u \Rightarrow 1+x = 2-u$

$\frac{du}{dx} = -1$

$dx = -du$.

$\int_2^1 \frac{2-u}{\sqrt{u}} -du$.

$-\int_2^1 (2-u)u^{-\frac{1}{2}}\,du$.

$-\int_2^1 (2u^{-\frac{1}{2}} - u^{\frac{1}{2}})\,du$

$= -\left[\frac{2u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^1$

$-\left[4u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_2^1$

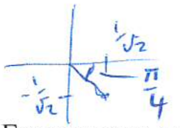
$-\left[(4-\frac{2}{3}) - (4\sqrt{2} - \frac{2}{3}\times 2\sqrt{2})\right]$

$-\left[(\frac{10}{3}) - (\frac{8}{3}\sqrt{2})\right]$

$\frac{8\sqrt{2}}{3} - \frac{10}{3}$

$\frac{8\sqrt{2}-10}{3}$

Question 3 (3 marks)



$r = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1}$ $\hookrightarrow \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = cis(-\frac{\pi}{4})$.

Find the cube roots of $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. Express your answers in polar form using principal values of the argument.

$z^3 = cis(-\frac{\pi}{4}) = cis(-\frac{\pi}{4} + 2k\pi)$

$z = cis\frac{1}{3}(-\frac{\pi}{4} + 2k\pi)$

$= cis(-\frac{\pi}{12} + \frac{2}{3}k\pi) = cis(-\frac{\pi}{12} + \frac{8k\pi}{12})$

$k = -1$: $z = cis(-\frac{\pi}{12} - \frac{8}{12}\pi) = cis(-\frac{9\pi}{12}) = cis(-\frac{3\pi}{4})$.

$k = 0$: $z = cis(-\frac{\pi}{12})$

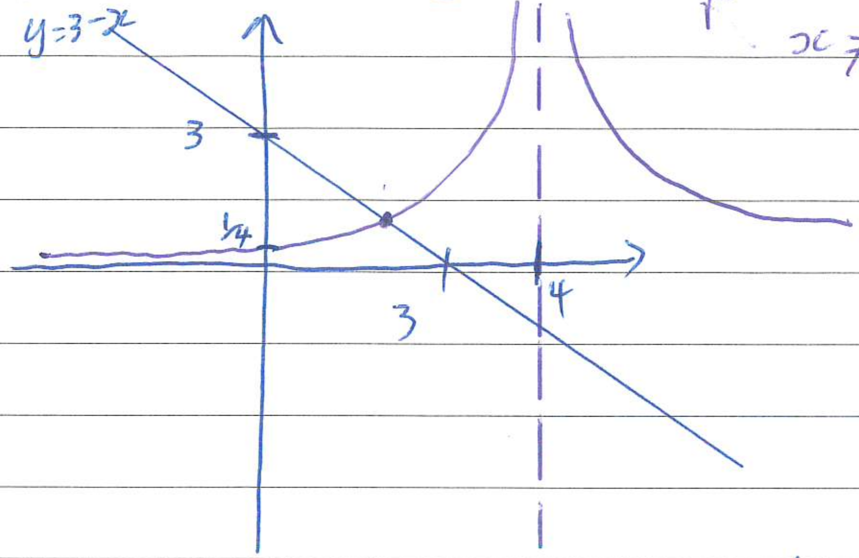
$k = 1$: $z = cis(-\frac{\pi}{12} + \frac{8\pi}{12}) = cis(\frac{7\pi}{12})$

Question 4 (4 marks)

Solve the inequality $3 - x > \frac{1}{|x-4|}$ for x, expressing your answer in interval notation.

Sketch $y = 3 - x$ and $y = \frac{1}{|x-4|}$

$x \neq 4$.



Intersection when $x < 3$. $\rightarrow y = \frac{1}{|x-4|}$

Can be written $y = \frac{1}{4-x}$.

$$3 - x = \frac{1}{4-x}$$

$$(3-x)(4-x) = 1$$

$$12 - 7x + x^2 = 1$$

$$x^2 - 7x + 11 = 0.$$

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4 \times 1 \times 11}}{2}$$

$$= \frac{7 \pm \sqrt{49 - 44}}{2}$$

$$= \frac{7 \pm \sqrt{5}}{2}.$$

From graph.

$$x = \frac{7 - \sqrt{5}}{2}.$$

Solution

$$x \in \left(-\infty, \frac{7-\sqrt{5}}{2}\right).$$

Question 5 (4 marks)

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}$, where m is an integer.

The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $-\frac{11}{18}\left(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}\right)$.

a. Find the value of m. 3 marks

$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} =$ $\quad |\underset{\sim}{b}| = \sqrt{1^2 + m^2 + (-1)^2} = \sqrt{2+m^2}$

$(2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}) \cdot \frac{1}{\sqrt{2+m^2}}(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}) = \frac{2 - 3m - 1}{\sqrt{2+m^2}} = \frac{1-3m}{\sqrt{2+m^2}}$

$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{1-3m}{\sqrt{2+m^2}} \times \frac{1}{\sqrt{2+m^2}}(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}) = -\frac{11}{8}(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k})$

$1{,}8 \times 3 = 54$

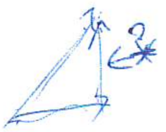
$\Rightarrow \frac{1-3m}{2+m^2} = -\frac{11}{18}$ $\quad \Rightarrow m = \frac{10}{11}$ or $m = 4$

$18 - 54m = -22 - 11m^2$ $\quad$ ~~m is an integer~~

$11m^2 - 54m + 40 = 0$

$(11m - 10)(m - 4) = 0$ $\quad \therefore m = 4.$

b. Find the component of $\underset{\sim}{a}$ that is perpendicular to $\underset{\sim}{b}$. 1 mark



$\underset{\sim}{a}$ in dir^n $\underset{\sim}{b} = -\frac{11}{18}(\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$.

Want. $\underset{\sim}{a} - \left(-\frac{11}{18}(\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})\right)$

$(2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}) + \left(\frac{11}{18}\underset{\sim}{i} + \frac{44}{18}\underset{\sim}{j} - \frac{11}{18}\underset{\sim}{k}\right)$

$= \frac{47}{18}\underset{\sim}{i} - \frac{10}{18}\underset{\sim}{j} + \frac{7}{18}\underset{\sim}{k}$

Question 6 (5 marks)

Let $f(x) = \arctan(3x - 6) + \pi$.

a. Show that $f'(x) = \dfrac{3}{9x^2 - 36x + 37}$. 1 mark

$$f'(x) = \frac{1}{1+(3x-6)^2}$$

$$= \frac{1}{1+9x^2-18x-18x+36}$$

$$f'(x) = \frac{1}{9x^2-36x+37}.$$

b. Hence, show that the graph of f has a point of inflection at $x = 2$. 2 marks

$$f'(2) = \frac{1}{36-72+37} = \frac{1}{1} = 1.$$

$$f'(1) = \frac{1}{9-36+37} = \frac{1}{10}$$

$$f'(3) = \frac{1}{81-108+37} = \frac{1}{10}$$

x	1	2	3
$f'(x)$	$\frac{1}{10}$	1	$\frac{1}{10}$

3.6, 3, 108

81, 37, 118

Gradient +ve each side $\Rightarrow$ Inflection

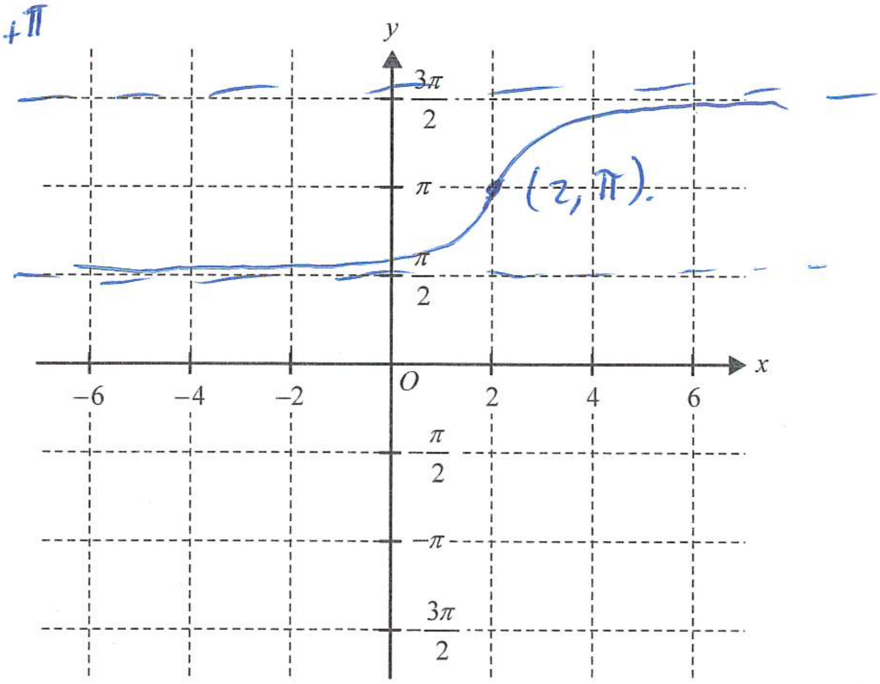
VCAA Answer. $f''(x) = \dfrac{-54(x-2)}{(9x^2-36x+37)^2}$ $\quad f''(2) = \dfrac{-54(2-2)}{(\quad)^2} = 0$

$\Rightarrow$ Point of Inflection.

c. Sketch the graph of $y = f(x)$ on the axes provided below. Label any asymptotes with their equations and the point of inflection with its coordinates. 2 marks

$\arctan(3x-6)+\pi$

Trans $\uparrow$ π



$f(x)\ \frac{4}{1+x^2} = 4(1+x^2)^{-1} \to f'(x) = -1 \times 4(1+x^2)^{-2} \times 2x$

$= -\frac{8x}{(1+x^2)^2}$

Question 7 (5 marks)

Consider the function defined by

$$f(x)=\begin{cases} mx+n, & x<1 \\ \dfrac{4}{1+x^2}, & x\geq 1 \end{cases}$$

$$f'(x) = \begin{cases} m \\ -\dfrac{8x}{(1+x^2)^2} \end{cases}$$

where m and n are real numbers.

a. Given that $f(x)$ and $f'(x)$ are continuous over R, show that $m = -2$ and $n = 4$. 2 marks

When $x = 1$

$m+n = \frac{4}{1+1^2}$ $\qquad m = \frac{-8\times 1}{(1+1^2)^2}$

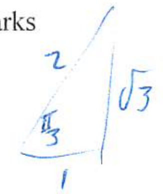
$m+n = 2.$ $\qquad m = \frac{-8}{4}$

$m = -2.$

$-2+n = 2$

$n = 4.$

b. Find the area enclosed by the graph of the function, the x-axis and the lines $x = 0$ and $x = \sqrt{3}$. 3 marks



$\text{Area} = \int_0^1 -2x+4\,dx + \int_1^{\sqrt{3}} \frac{4}{1+x^2}\,dx.$

$= \left[-x^2+4x\right]_0^1 + \left[4\,\text{Tan}^{-1}(x)\right]_1^{\sqrt{3}}$

$= \left[(-(1)^2+4\times 1)-(-(0)^2+4\times 0)\right] + \left[(4\,\text{Tan}^{-1}(\sqrt{3}))-(4\,\text{Tan}^{-1}(1))\right]$

$= \left[(-1+4)-0\right] + \left[(4\times\frac{\pi}{3})-(4\times\frac{\pi}{4})\right]$

$= 3 + \frac{4\pi}{3} - \pi$

$= 3 + \frac{\pi}{3}.$

Question 8 (5 marks)

Find the volume, V, of the solid of revolution formed when the graph of $y = 2\sqrt{\frac{x^2+x+1}{(x+1)(x^2+1)}}$ is rotated about the x-axis over the interval $\left[0, \sqrt{3}\right]$. Give your answer in the form $V = 2\pi\left(\log_e(a) + b\right)$, where a, $b \in R$.

$$V = \int_0^{\sqrt{3}} \pi y^2 \, dx = \int_0^{\sqrt{3}} \pi \left(2\sqrt{\frac{x^2+x+1}{(x+1)(x^2+1)}}\right)^2 dx$$

$$= 4\pi \int_0^{\sqrt{3}} \frac{x^2+x+1}{(x+1)(x^2+1)} \, dx$$

$$= 4\pi \times \frac{1}{2} \int_0^{\sqrt{3}} \frac{1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1} \, dx$$

$$= 2\pi \left[\log_e(x+1) + \frac{1}{2}\log_e(x^2+1) + \tan^{-1}(x) \right]_0^{\sqrt{3}}$$

$$= 2\pi \left[\left(\log_e(\sqrt{3}+1) + \frac{1}{2}\log_e(3+1) + \tan^{-1}(\sqrt{3})\right) - \left(\log_e(1) + \frac{1}{2}\log_e(1) + \tan^{-1}(0)\right) \right]$$

$$= 2\pi \left[\log_e(\sqrt{3}+1) + \log_e(4)^{\frac{1}{2}} + \frac{\pi}{3} \right]$$

$$= 2\pi \left[\log_e(2\sqrt{3}+2) + \frac{\pi}{3} \right]$$

$$\frac{x^2+x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{1}{2(x+1)} + \frac{1(x+1)}{2(x^2+1)} = \frac{1}{2}\left[\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\right]$$

$$= \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)}.$$

$$x^2+x+1 = A(x^2+1) + (Bx+C)(x+1).$$

When $x = -1$.

$$1-1+1 = A(1+1) + (B+C)(-1+1)$$

$$1 = 2A$$

$$A = \frac{1}{2}.$$

$$x^2+x+1 = \frac{1}{2}(x^2+1) + Bx^2+Bx+Cx+C.$$

$$x^2+x+1 = \frac{1}{2}x^2 + \frac{1}{2} + Bx^2 + Bx + Cx + C.$$

x^2 coefficients

$$1 = \frac{1}{2} + B$$

$$B = \frac{1}{2}.$$

Constant.

$$1 = \frac{1}{2} + C$$

$$C = \frac{1}{2}.$$

Question 9 (5 marks)

Consider the curve defined parametrically by

$$x = \arcsin(t)$$

$$y = \log_e(1+t) + \frac{1}{4}\log_e(1-t)$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}}$$

where $t \in [0, 1)$.

a. $\left(\frac{dy}{dt}\right)^2$ can be written in the form $\frac{1}{a(1+t)^2} + \frac{1}{b(1-t^2)} + \frac{1}{c(1-t)^2}$, where a, b and c are real numbers.

Show that $a = 1$, $b = -2$ and $c = 16$. 2 marks

$$\frac{dy}{dt} = \frac{1}{1+t} - \frac{1}{4(1-t)} \quad \left(\frac{dy}{dt}\right)^2 = \left(\frac{1}{1+t} - \frac{1}{4(1-t)}\right)^2$$

$$= \frac{1}{(1+t)^2} - 2 \times \frac{1}{4(1+t)(1-t)} + \frac{1}{16(1-t)^2}$$

$$= \frac{1}{(1+t)^2} - \frac{1}{2(1-t^2)} + \frac{1}{16(1-t)^2}$$

$$\Rightarrow a = 1 \quad b = -2 \quad c = 16.$$

b. Find the arc length, s, of the curve from $t = 0$ to $t = \frac{1}{2}$. Give your answer in the form $s = \log_e(m) + n\log_e(p)$, where $m, n, p \in Q$. 3 marks

$$S = \int_0^{\frac{1}{2}} \sqrt{\left(\frac{1}{\sqrt{1-t^2}}\right)^2 + \left(\frac{1}{(1+t)^2} - \frac{1}{2(1-t^2)} + \frac{1}{16(1-t)^2}\right)} \, dt$$

$$= \int_0^{\frac{1}{2}} \sqrt{\frac{1}{1-t^2} + \frac{1}{(1+t)^2} - \frac{1}{2(1-t^2)} + \frac{1}{16(1-t)^2}} \, dt$$

$$= \int_0^{\frac{1}{2}} \sqrt{\frac{1}{(1+t)^2} + \frac{1}{2}\frac{1}{(1-t^2)} + \frac{1}{16(1-t)^2}} \, dt$$

$$= \int_0^{\frac{1}{2}} \sqrt{\left(\frac{1}{1+t} + \frac{1}{4}\frac{1}{(1-t)}\right)^2} \, dt$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1+t} + \frac{1}{4(1-t)} \, dt.$$

$$= \left[\log_e(1+t) - \frac{1}{4}\log_e(1-t)\right]_0^{\frac{1}{2}} = \left(\log_e\left(1+\tfrac{1}{2}\right) - \tfrac{1}{4}\log_e\left(1-\tfrac{1}{2}\right) - 0\right)$$

$$= \log_e\left(\tfrac{3}{2}\right) - \tfrac{1}{4}\log_e\left(\tfrac{1}{2}\right)$$

END OF QUESTION AND ANSWER BOOK

$$= \log_e\left(\tfrac{3}{2}\right) + \tfrac{1}{4}\log_e(2)$$