VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2020

ANB

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER

SPECIALIST MATHEMATICS

Written examination 2

Friday 20 November 2020

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1

The y-intercept of the graph of $y = f(x)$, where $f(x) = \dfrac{(x-a)(x+3)}{(x-2)}$, is also a stationary point when a equals

A. -2

B. $-\dfrac{6}{5}$

C. 0

D. $\dfrac{6}{5}$

E. 2

$f(0) = \dfrac{-a \times 3}{-2} = \dfrac{3}{2}a$

Not useful

$f'(x) = \dfrac{x^2 - 4x + 5a - 6}{(x-2)^2}$

Use Calculator

$f'(0) = 0 = \dfrac{5a-6}{4}$

$5a - 6 = 0$

$a = \dfrac{6}{5}$

Question 2

A function f has the rule $f(x) = |b\cos^{-1}(x) - a|$, where $a > 0$, $b > 0$ and $a < \dfrac{b\pi}{2}$.

The range of f is

A. $[-a, b\pi - a]$

B. $[0, b\pi - a]$

C. $[a, b\pi - a]$

D. $[0, b\pi + a]$

E. $[a - b\pi, a]$

Absolute Value.

$\cos^{-1}(x)$ Range $[0, \pi]$

$b\cos^{-1}(x)$ Range $[0, b\pi]$

$b\cos^{-1}(x) - a$ Range $[-a, b\pi - a]$

but $a < \dfrac{b\pi}{2}$ will be +ve.

$|b\cos^{-1}(x) - a|$ Range $[0, b\pi - a]$

Question 3

A train is travelling from Station A to Station B. The train starts from rest at Station A and travels with constant acceleration for 30 seconds until it reaches a velocity of 10 ms^{-1}. It then travels at this velocity for 200 seconds. The train then slows down, with constant acceleration, and stops at Station B having travelled for 260 seconds in total. Let v ms^{-1} be the velocity of the train at time t seconds.

The velocity v as a function of t is given by

A. $v(t) = \begin{cases} \frac{1}{3}t, & 0 \le t \le 30 \\ 10, & 30 < t \le 230 \\ \frac{1}{3}(260 - t), & 230 < t \le 260 \end{cases}$

B. $v(t) = \begin{cases} \frac{1}{3}t, & 0 \le t \le 30 \\ 10, & 30 < t \le 230 \\ \frac{1}{3}(230 - t), & 230 < t \le 260 \end{cases}$

C. $v(t) = \begin{cases} 3t, & 0 \le t \le 30 \\ 10, & 30 < t \le 230 \\ 3(230 - t), & 230 < t \le 260 \end{cases}$

D. $v(t) = \begin{cases} 3t, & 0 \le t \le 30 \\ 10, & 30 < t \le 230 \\ 3(260 - t), & 230 < t \le 260 \end{cases}$

E. $v(t) = \begin{cases} \frac{1}{3}t, & 0 \le t \le 30 \\ 10, & 30 < t \le 200 \\ \frac{1}{3}(230 - t), & 200 < t \le 230 \end{cases}$

$v = 10$ when $t = 30$

Const acc^n $\Rightarrow v = at.$

$10 = a \times 30$

$a = \frac{1}{3}.$

First part $\frac{1}{3}t \to 0 \le t \le 30$

Second Part

$10 \to 30 < t \le 230$

3rd Part const acc^n. another 30 seconds

$a = \frac{\Delta v}{t} = \frac{\cancel{0} - 10}{30}$

$= -\frac{1}{3}.$

for $t = 230$ $v = 10$

$\Rightarrow \frac{1}{3}(? - 230) = 10$

$? - 230 = 30$

$? = 260.$

$\Rightarrow A.$

Question 4

Let $f(x) = \dfrac{\sqrt{x-1}}{x}$ over its implied domain and $g(x) = \operatorname{cosec}^2 x$ for $0 < x < \dfrac{\pi}{2}$.

The rule for $f(g(x))$ and the range, respectively, are given by

A. $f(g(x)) = \operatorname{cosec}^2\left(\dfrac{\sqrt{x-1}}{x}\right)$, $[1, \infty)$

B. $f(g(x)) = \operatorname{cosec}^2\left(\dfrac{\sqrt{x-1}}{x}\right)$, $[2, \infty)$

C. $f(g(x)) = \sin(x)\cos(x)$, $[-0.5, 0.5]\backslash\{0\}$

D. $f(g(x)) = \sin(x)\cos(x)$, $\left(0, \dfrac{1}{2}\right)$

E. $f(g(x)) = \dfrac{1}{2}\sin(2x)$, $\left(0, \dfrac{1}{2}\right]$

$f(g(x)) = \dfrac{\sqrt{\operatorname{cosec}^2 x - 1}}{\operatorname{cosec}^2 x}$

$= \dfrac{\sqrt{\cot^2 x}}{\operatorname{cosec}^2 x}$

$= \dfrac{\cot x}{\operatorname{cosec}^2 x}$

$= \dfrac{\cos x}{\sin x} \times \dfrac{\sin^2 x}{1}$

$= \sin x \cos x = \frac{1}{2}\sin(2x).$

SECTION A – continued

TURN OVER

Question 5

Given the complex number $z = a + bi$, where $a \in R\backslash\{0\}$ and $b \in R$, $\dfrac{4z\bar{z}}{(z+\bar{z})^2}$ is equivalent to

A. $1+\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$

B. $4[\text{Re}(z)\times\text{Im}(z)]$

C. $4\left([\text{Re}(z)]^2+[\text{Im}(z)]^2\right)$

D. $4\left[1+(\text{Re}(z)+\text{Im}(z))^2\right]$

E. $\dfrac{2\times\text{Im}(z)}{[\text{Re}(z)]^2}$

$\dfrac{4(a+bi)(a-bi)}{(a+bi+a-bi)^2}$

$\dfrac{4(a^2+b^2)}{(2a)^2}$

$\dfrac{4(a^2+b^2)}{4a^2} = \dfrac{a^2}{a^2}+\dfrac{b^2}{a^2}$

$= 1+\left(\dfrac{b}{a}\right)^2$

Question 6

For the complex polynomial $P(z) = z^3 + az^2 + bz + c$ with real coefficients a, b and c, $P(-2) = 0$ and $P(3i) = 0$.
The values of a, b and c are respectively

A. $-2, 9, -18$

B. $3, 4, 12$

C. $2, 9, 18$

D. $-3, -4, 12$

E. $2, -9, -18$

$P(-2)=0 \quad 0=(-2)^3+a(-2)^2+b\times(-2)+c.$ } long way
$P(3i)=0 \quad 0=(3i)^3+a(3i)^2+b\times(3i)+c.$ } around

conjugate Root Theorem says $P(-3i)=0$.
Thus factors are $(x+2)(x-3i)(x+3i)$
get C from $2\times -3i\times 3i = -18i^2 = 18$
only one option

Question 7

For non-zero real constants a and b, where $b < 0$, the expression $\dfrac{1}{ax(x^2+b)}$ in partial fraction form with linear denominators, where A, B and C are real constants, is

A. $\dfrac{A}{ax}+\dfrac{Bx+C}{x^2+b}$ b is -ve so can't have.

B. $\dfrac{A}{ax}+\dfrac{B}{x+\sqrt{b}}+\dfrac{C}{x-\sqrt{b}}$

C. $\dfrac{A}{x}+\dfrac{B}{ax+\sqrt{|b|}}+\dfrac{C}{ax-\sqrt{|b|}}$

D. $\dfrac{A}{x}+\dfrac{B}{x+\sqrt{|b|}}+\dfrac{C}{x-\sqrt{|b|}}$

E. $\dfrac{A}{ax}+\dfrac{B}{(x+\sqrt{b})^2}+\dfrac{C}{x+\sqrt{b}}$

$\dfrac{1}{ax(x^2+b)} = \dfrac{A}{ax}+\dfrac{Bx+C}{x^2+b}$ ← Not linear

but $b<0$
x^2+b effectively $x^2-(-b)$
factorise to $(x+\sqrt{|b|})(x-\sqrt{|b|})$
$\Rightarrow$ C or D.
$\dfrac{A}{ax}$ can be written as a single as they are constants
$\Rightarrow$ D.

Question 8

Given that $(x + iy)^{14} = a + ib$, where $x, y, a, b \in R$, $(y - ix)^{14}$ for all values of x and y is equal to

A. $-a - ib$

B. $b - ia$

C. $-b + ia$

D. $-a + ib$

E. $b + ia$

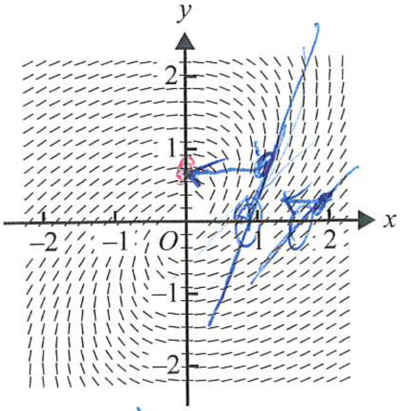
$$(y - ix)^{14} = \left[-i(iy + x)\right]^{14}$$
$$= (-i)^{14}(x + iy)^{14}$$
$$= ((-i)^2)^7(x + iy)^{14}$$
$$= (-1)^7(x + iy)^{14}$$
$$= -1(x + iy)^{14}$$
$$= -1(a + ib)$$
$$= -a - ib.$$

Question 9

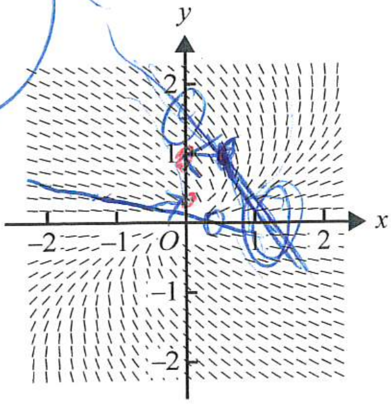
$P(x, y)$ is a point on a curve. The x-intercept of a tangent to point $P(x, y)$ is equal to the y-value at P.

Which one of the following slope fields best represents this curve?

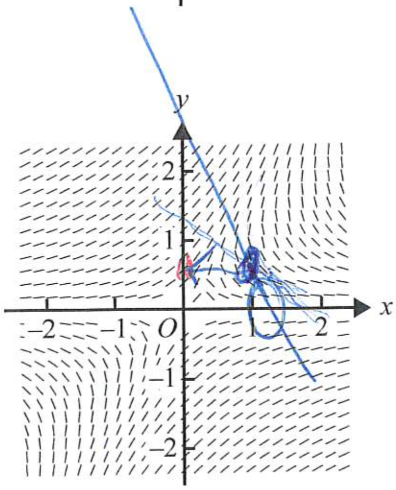
A.



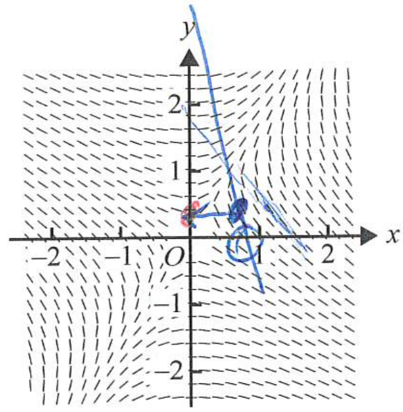
B.



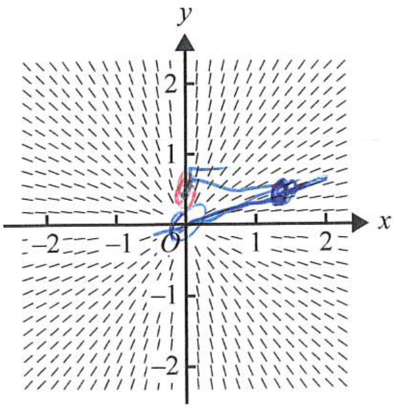
C.



D.



E.



x int of tangent
y value at p
must be the same.
B only Possibility.

Question 10

A tank initially contains 300 grams of salt that is dissolved in 50 L of water. A solution containing 15 grams of salt per litre of water is poured into the tank at a rate of 2 L per minute and the mixture in the tank is kept well stirred. At the same time, 5 L of the mixture flows out of the tank per minute.

A differential equation representing the mass, m grams, of salt in the tank at time t minutes, for a non-zero volume of mixture, is

A. $\frac{dm}{dt}=0$

B. $\frac{dm}{dt}=-\frac{5m}{50-5t}$

C. $\frac{dm}{dt}=30-\frac{m}{10}$

D. $\frac{dm}{dt}=30-\frac{5m}{50-3t}$

E. $\frac{dm}{dt}=30-\frac{5m}{50-5t}$

Volume = $50+2t-5t$

Volume = $50-3t$.

Each second going out is 5 x Concentration

→ $5 \times \frac{M}{50-3t}$ ← Mass ← Volume

going in is $15 \times 2 = 30$

$\frac{dm}{dt} = 30 - \frac{5m}{50-3t}$

Question 11

With a suitable substitution $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(x)}{\sec^2(x)-3\tan(x)+1}\,dx$ can be expressed as

~~A.~~ $\int_1^{\frac{1}{\sqrt{3}}}\left(\frac{1}{u-1}-\frac{1}{u-2}\right)du$

~~B.~~ $\int_1^{\sqrt{3}}\left(\frac{1}{3(u-3)}-\frac{1}{3u}\right)du$

C. $\int_1^{\sqrt{3}}\left(\frac{1}{u-2}-\frac{1}{u-1}\right)du$

D. $\int_1^{\sqrt{3}}\left(\frac{1}{u-1}-\frac{1}{u-2}\right)du$

~~E.~~ $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{1}{3(u-1)}-\frac{1}{3(u+2)}\right)du$

$u = \tan(x)$. $\sec^2(x) = 1+\tan^2(x)$

$\frac{du}{dx} = \sec^2(x)$.

$\int \frac{\frac{du}{dx}}{1+u^2-3u+1}\,dx$

$\int \frac{1}{u^2-3u+2}\,du$.

$\int \frac{1}{(u-2)(u-1)}\,du$ → B + E out.

$\frac{1}{(u-2)(u-1)} \equiv \frac{A}{u-2}+\frac{B}{u-1}$

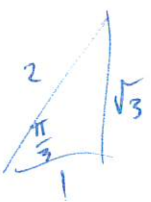
$\equiv \frac{A(u-1)+B(u-2)}{(u-2)(u-1)}$

$u=1$

$1=-B$.

$B=-1$

∴ C



$x=\frac{\pi}{3}$ $\quad$ $x=\frac{\pi}{4}$

$u=\tan\left(\frac{\pi}{3}\right)$ $\quad$ $u=\tan\left(\frac{\pi}{4}\right)$

$u=\sqrt{3}$ $\quad$ $=1$

A+E out.

on formula sheet.

Question 12

$h = 0.1$

If $\dfrac{dy}{dx} = e^{\cos(x)}$ and $y_0 = e$ when $x_0 = 0$, then, using Euler's formula with step size 0.1, y_3 is equal to

A. $e + 0.1\left(1 + e^{\cos(0.1)}\right)$

B. $e + 0.1\left(1 + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$

C. $e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$

D. $e + 0.1\left(e^{\cos(0.1)} + e^{\cos(0.2)} + e^{\cos(0.3)}\right)$

E. $e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)} + e^{\cos(0.3)}\right)$

$x_0 = 0$ $\quad y_0 = e$

$x_1 = 0 + 0.1 = 0.1$ $\quad y_1 = e + 0.1 \times e^{\cos(0)} = e + 0.1e$

$x_2 = 0.1 + 0.1 = 0.2.$ $\quad y_2 = e + 0.1e + 0.1 \times e^{\cos(0.1)} = e + 0.1\left(e + e^{\cos(0.1)}\right)$

one more → looks like C

Question 13

The vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \lambda\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{k}$ will be **linearly dependent** when the value of λ is

A. 1

B. 2

C. 3

D. ~~4~~

E. 5

$m\underset{\sim}{a} + n\underset{\sim}{c} = \underset{\sim}{b}.$

$m(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) + n(\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{k}) = \lambda\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{i}$ components: $m + n = \lambda$, $\frac{3}{2} + \frac{7}{2} = \lambda$, $\frac{10}{2} = \lambda$

$\underset{\sim}{j}$: $2m = 3$, $m = \frac{3}{2}.$

$\underset{\sim}{k}$: $-m + n = 2.$ $n = 2 + \frac{3}{2} = \frac{7}{2}.$

Question 14

The magnitude of the component of the force $\underset{\sim}{F} = \underset{\sim}{i} + 6\underset{\sim}{j} - 18\underset{\sim}{k}$ that acts in the direction $\underset{\sim}{d} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 6\underset{\sim}{k}$ is

A. $\dfrac{92}{19}$

B. $\dfrac{92}{7}$

C. $\dfrac{124}{7}$

D. $\dfrac{92}{11}$

E. $\dfrac{18}{7}$

for Magnitude.

$\underset{\sim}{F}$ on $\underset{\sim}{d}$. $= \underset{\sim}{F} \cdot \hat{\underset{\sim}{d}}$ $\quad |d| = \sqrt{4+9+36} = \sqrt{49} = 7$

$= (\underset{\sim}{i} + 6\underset{\sim}{j} - 18\underset{\sim}{k}) \cdot \frac{1}{7}(2\underset{\sim}{i} - 3\underset{\sim}{j} - 6\underset{\sim}{k})$

$= \frac{1}{7}\left(1 \times 2 + 6 \times -3 + (-18) \times (-6)\right)$

$= \frac{1}{7}\left(2 - 18 + 108\right)$

$= \dfrac{92}{7}$

Question 15

Two forces, $\underset{\sim}{F}_A = 4\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{F}_B = 2\underset{\sim}{i} + 5\underset{\sim}{j}$, act on a particle of mass 3 kg. The particle is initially at rest at position $\underset{\sim}{i} + \underset{\sim}{j}$. All force components are measured in newtons and displacements are measured in metres.

The cartesian equation of the path of the particle is

$R = F_A + F_B = 6i + 3j$

$a = \frac{R}{m} = \frac{6i+3j}{3} = 2i + j$

$v = \int a\,dt = 2t\,i + t\,j + c$

$t = 0 \quad v = 0 \Rightarrow c = 0$

$r = t^2 i + \frac{t^2}{2} j + c. \quad t=0 \ r = i + j$

$r = (t^2+1)\,i + (\frac{t^2}{2}+1)\,j.$

$x = t^2 + 1$

$t^2 = x - 1$

$y = \frac{x-1}{2} + 1$

$= \frac{x+1}{2}$

A. $y = \frac{x}{2}$

B. $y = \frac{x}{2} - \frac{1}{2}$

C. $y = \frac{(x+1)^2}{2} + 1$

D. $y = \frac{(x-1)^2}{2} + 1$

E. $y = \frac{x}{2} + \frac{1}{2}$

Question 16

Let $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$, where the acute angle between these vectors is θ.

The value of $\sin(2\theta)$ is

Sin and Cos +ve

$a \cdot b = |a||b| \cos\theta$

$2 - 8 + 8 = \sqrt{1+4+4} \times \sqrt{4+16+16} \cos\theta$

$2 = \sqrt{9} \times \sqrt{36} \cos\theta$

$2 = 3 \times 6 . \cos\theta$

$\cos\theta = \frac{2}{18} = \frac{1}{9}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \frac{1}{81} = 1$

$\sin^2\theta = \frac{80}{81}$

$\sin\theta = \frac{\sqrt{80}}{9} = \frac{4\sqrt{5}}{9}$

$\sin(2\theta) = 2\sin\theta\cos\theta = 2 \times \frac{4\sqrt{5}}{9} \times \frac{1}{9} = \frac{8\sqrt{5}}{81}$

A. $\frac{1}{9}$

B. $\frac{4\sqrt{5}}{9}$

C. $\frac{4\sqrt{5}}{81}$

D. $\frac{8\sqrt{5}}{81}$

E. $\frac{2\sqrt{46}}{25}$

Question 17

The velocity, v ms^{-1}, of a particle at time $t \geq 0$ seconds and at position $x \geq 1$ m from the origin is $v = \frac{1}{x}$.
The acceleration of the particle, in ms^{-2}, when $x = 2$ is

$v = \frac{1}{x}$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$a = \frac{d}{dx}\left(\frac{1}{2} \times \frac{1}{x^2}\right)$

$a = \frac{d}{dx}\left(\frac{1}{2x^2}\right)$

$a = -\frac{1}{x^3}$

$x = 2$

$a = -\frac{1}{2^3}$

$= -1/8.$

A. $-\frac{1}{4}$

B. $-\frac{1}{8}$

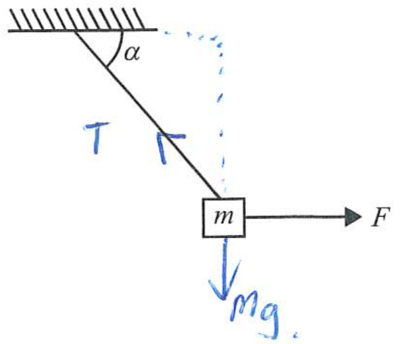
C. $\frac{1}{8}$

D. $\frac{1}{2}$

E. $\frac{1}{4}$

Question 18

A particle of mass m kilograms hangs from a string that is attached to a fixed point. The particle is acted on by a horizontal force of magnitude F newtons. The system is in equilibrium when the string makes an angle α to the horizontal, as shown in the diagram below. The tension in the string has magnitude T newtons.



Vertically

$T\sin\alpha = mg$.

$\sin\alpha = \frac{mg}{T}$

Horizontally

$T\cos\alpha = F$

$\cos\alpha = \frac{F}{T}$

$\tan\alpha = \frac{\frac{mg}{T}}{\frac{F}{T}} = \frac{Mg}{T} \times \frac{T}{F} = \frac{Mg}{F}$

The value of $\tan(\alpha)$ is

A. $\frac{mg}{T}$

B. $\frac{T}{mg}$

C. $\frac{T}{F}$

D. $\frac{F}{mg}$

E. $\frac{mg}{F}$

Question 19

A cricket ball of mass 0.02 kg, moving with velocity $2\underset{\sim}{i} - 10\underset{\sim}{j}$ ms^{-1}, is hit and after impact travels with velocity $2\underset{\sim}{i} - 7\underset{\sim}{j}$ ms^{-1}.

The magnitude of the change in momentum of the cricket ball, in kg ms^{-1}, is closest to

A. 0.04

B. 0.06

C. 0.10

D. 0.24

E. 0.34

$\Delta \underset{\sim}{p} = m\Delta\underset{\sim}{v} = m(v_2 - v_1)$

$= 0.02(2\underset{\sim}{i} - 7\underset{\sim}{j} - [2i - 10\underset{\sim}{j}])$

$= 0.02(3\underset{\sim}{j})$.

$= 0.06\underset{\sim}{j}$.

Question 20

An object of mass 2 kg is suspended from a spring balance that is inside a lift travelling downwards. If the reading on the spring balance is 30 N, the acceleration of the lift is

↑ +ve.

A. 5.2 ms^{-2} upwards.

B. 5.2 ms^{-2} downwards.

C. 9.8 ms^{-2} downwards.

D. 10.4 ms^{-2} upwards.

E. 10.4 ms^{-2} downwards.

$\underset{\sim}{R} = m\underset{\sim}{a}$

$30 - 2g = 2\underset{\sim}{a}$.

$\frac{30 - 2\times 9.8}{2} = \underset{\sim}{a}$

$\underset{\sim}{a} = 5.2$ direction up.

END OF SECTION A

CONTINUES OVER PAGE

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

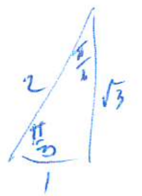
Question 1 (12 marks)

A particle moves in the x-y plane such that its position in terms of x and y metres at t seconds is given by the parametric equations

$$x = 2\sin(2t)$$
$$y = 3\cos(t)$$

where $t \geq 0$.

a. Find the distance, in metres, of the particle from the origin when $t = \frac{\pi}{6}$. 2 marks



$x = 2\sin\left(2 \times \frac{\pi}{6}\right)$ $\quad y = 3\cos\left(\frac{\pi}{6}\right)$ $\quad$ distance $= \sqrt{x^2 + y^2}$

$= 2\sin\left(\frac{\pi}{3}\right)$ $\quad = 3 \times \frac{\sqrt{3}}{2}$ $\quad = \sqrt{(\sqrt{3})^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}$

$= 2 \times \frac{\sqrt{3}}{2}$ $\quad = \frac{3\sqrt{3}}{2}$

$= \sqrt{3}$ $\quad = \frac{\sqrt{39}}{2}$.

b. **i.** Express $\frac{dy}{dx}$ in terms of t and, hence, find the equation of the tangent to the path of the particle at $t = \pi$ seconds. 3 marks

$\frac{dx}{dt} = 4\cos(2t)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $\quad \frac{dy}{dt} = -3\sin(t)$

when $t = \pi$

$= -3\sin(t) \times \frac{1}{4\cos(2t)}$ $\quad x = 2\sin(2\pi)$ $\quad y = 3\cos(\pi)$

$= 0$ $\quad = -3$

$\frac{dy}{dx} = \frac{-3\sin(t)}{4\cos(2t)}$ $\quad m = \frac{-3\sin(\pi)}{4\cos(2\pi)}$

$= 0$.

Tangent $y - {-3} = 0(x - 0)$

$y + 3 = 0$

$y = -3$.

ii. Find the velocity, $\underset{\sim}{v}$, in ms^{-1}, of the particle when $t = \pi$. 2 marks

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$ $\quad t = \pi$

$\underset{\sim}{r} = 2\sin(2t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}$ $\quad \underset{\sim}{v} = 4\cos(2\pi)\underset{\sim}{i} - 3\sin(\pi)\underset{\sim}{j}$

$\underset{\sim}{v} = 4\cos(2t)\underset{\sim}{i} - 3\sin(t)\underset{\sim}{j}$ $\quad = 4\underset{\sim}{i}$

iii. Find the magnitude of the acceleration, in ms^{-2}, when $t = \pi$. 2 marks

$\underset{\sim}{a} = -8\sin(2t)\underset{\sim}{i} - 3\cos(t)\underset{\sim}{j}$

$t = \pi$

$\underset{\sim}{a} = -8\sin(2\pi)\underset{\sim}{i} - 3\cos(\pi)\underset{\sim}{j} = -3\underset{\sim}{j}$ $\quad |\underset{\sim}{a}| = 3.$

c. Find the time, in seconds, when the particle first passes through the origin. 1 mark

$\underset{\sim}{r} = 2\sin(2t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}$

$2\sin(2t) = 0$ $\quad 3\cos(t) = 0.$ $\quad$ Calc.

First Time $\quad$ Solve$(2\sin(2t) = 3\cos(t), x) | 0 \leq x \leq 2\pi$

$t = \frac{\pi}{2}.$

d. Express the distance, d metres, travelled by the particle from $t = 0$ to $t = \frac{\pi}{6}$ as a definite integral and find this distance correct to three decimal places. 2 marks

arc length on formula sheet.

$d = \int_0^{\frac{\pi}{6}} \sqrt{(4\cos(2t))^2 + (-3\sin(t))^2}\, dt$

$= 1.80378$

$= 1.804$

Question 2 (11 marks)

Two complex numbers, u and v, are defined as $u = -2 - i$ and $v = -4 - 3i$.

a. Express the relation $|z - u| = |z - v|$ in the cartesian form $y = mx + c$, where $m, c \in R$. 3 marks

$z = x + yi$

$z - u = (x + yi) - (-2 - i)$

$= (x+2) + (y+1)i$

$|z - u| = \sqrt{(x+2)^2 + (y+1)^2}$

$z - v = (x+4) + (y+3)i$

$|z - v| = \sqrt{(x+4)^2 + (y+3)^2}$

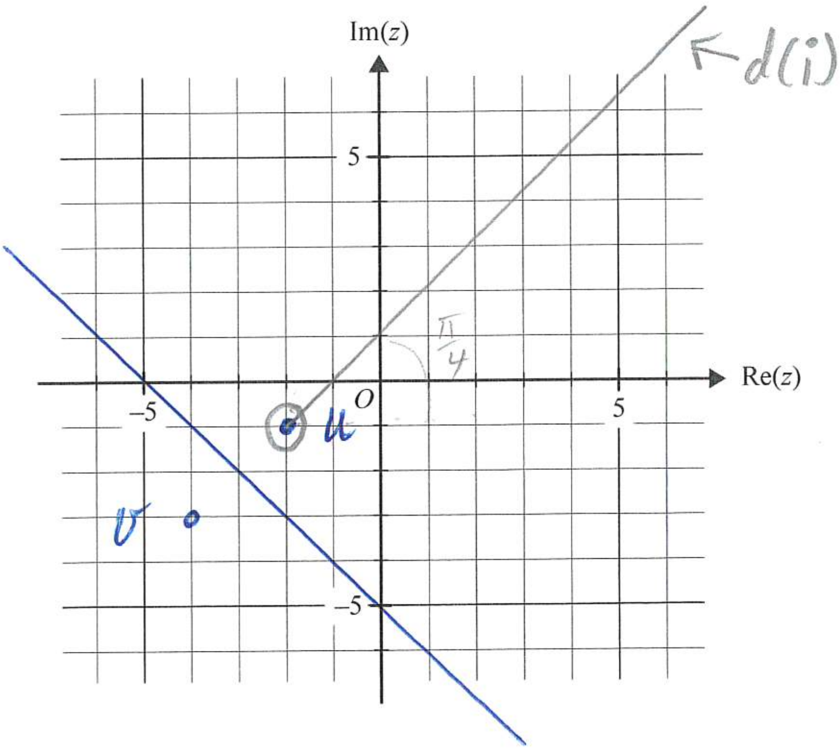
$\sqrt{(x+2)^2 + (y+1)^2} = \sqrt{(x+4)^2 + (y+3)^2}$

$(x+2)^2 + (y+1)^2 = (x+4)^2 + (y+3)^2$

Solve(, y) on Calc

$y = -x - 5$

b. Plot the points that represent u and v and the relation $|z - u| = |z - v|$ on the Argand diagram below. 2 marks



c. State a geometrical interpretation of the graph of $|z - u| = |z - v|$ in relation to the points that represent u and v. 1 mark

Perpendicular Bisector of the line joining the points u and v.

Ray translated to start at u

d. **i.** Sketch the ray given by $\text{Arg}(z-u)=\frac{\pi}{4}$ on the Argand diagram in **part b.** 1 mark

Remember O at the starting point of a ray.

ii. Write down the function that describes the ray $\text{Arg}(z-u)=\frac{\pi}{4}$, giving the rule in cartesian form. 1 mark

Domain $(-2,\infty)$ $f(x)=x+1$

$f:(-2,\infty)\rightarrow R: f(x)=x+1$

e. The points representing u and v and $-5i$ lie on the circle given by $|z-z_c|=r$, where z_c is the centre of the circle and r is the radius.

Find z_c in the form $a+ib$, where $a, b \in R$, and find the radius r. 3 marks

$|u-z_c|=|v-z_c|=|-5i-z_c|=r.$

$|(-2-a)+(-b-1)i|=|(-4-a)+(-b-3)i|=|(-a)+(-b-5)i|$

$(-2-a)^2+(-b-1)^2=(-4-a)^2+(-b-3)^2.$ Calc Sim Eq^{ns}.

and $(-2-a)^2+(-b-1)^2=(-a)^2+(-b-5)^2.$

$a=-5/3$ $b=-\frac{10}{3}$

$z_c=-\frac{5}{3}-\frac{10}{3}i$

$r=\sqrt{\left(-\frac{5}{3}\right)^2+\left(-\frac{10}{3}\right)^2}=\frac{5\sqrt{5}}{3}$

Care needs to be taken. Easy to loose things.

Question 3 (10 marks)

Let $f(x) = x^2 e^{-x}$.

a. Find an expression for $f'(x)$ and state the coordinates of the stationary points of $f(x)$. 2 marks

$f'(x) = -(x^2-2x)e^{-x}$

$f'(x)=0 = -(x^2-2x)e^{-x}$

$x^2-2x=0 \qquad e^{-x}=0$

No sol^n

$x(x-2)=0$

$x=0 \quad x=2 \quad$ Points $(0,0), \left(2, \frac{4}{e^2}\right)$

b. State the equation(s) of any asymptotes of $f(x)$. 1 mark

$y=0$ Graph on Calc.

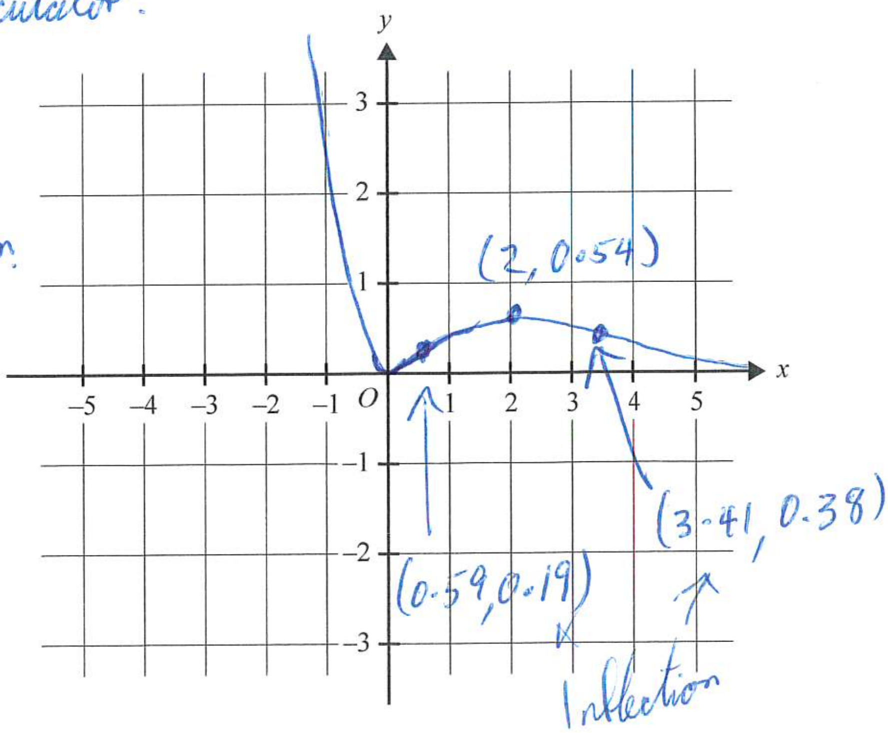
c. Sketch the graph of $y = f(x)$ on the axes provided below, labelling the local maximum stationary point and all points of inflection with their coordinates, correct to two decimal places. 3 marks

Use Calculator.

G-Solve

For Max.

and Inflection



Let $g(x) = x^n e^{-x}$, where $n \in Z$. ← Calculator.

d. Write down an expression for $g''(x)$. 1 mark

$x^{n-2}(x^2+n^2-2nx-n)e^{-x}$

$x^{n-2}[(x-n)^2-n]e^{-x}$

Note: Any correct variation was accepted

e. **i.** Find the non-zero values of x for which $g''(x) = 0$. 1 mark

$x^{n-2}(x^2+n^2-2nx-n)e^{-x} = 0$

$x \neq 0$ $x = n-\sqrt{n}$, $x = n+\sqrt{n}$

ii. Complete the following table by stating the value(s) of n for which the graph of $g(x)$ has the given number of points of inflection. 2 marks

Number of points of inflection	Value(s) of n (where $n \in Z$)
0	$n \leq 0$
1	$n = 1$
2	$n = 2, 4, 6, \ldots$
3	$n = 3, 5, 7, \ldots$

← No Solution

$\leftarrow n - \sqrt{n} = 0$ when $n = 1$

$x^{n-2} \rightarrow$ odd $x^{n-2} \rightarrow x^{odd} \rightarrow$ Inflection

$x^{n-2} \rightarrow$ even $x^{n-2} \rightarrow x^{even} \rightarrow$ Turning Point

Question 4 (14 marks)

A pilot is performing at an air show. The position of her aeroplane at time t relative to a fixed origin O is given by $\underset{\sim}{r}_A(t) = \left(450 - 150\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{i} + \left(400 - 200\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a unit vector in a horizontal direction, $\underset{\sim}{j}$ is a unit vector vertically up, displacement components are measured in metres and time t is measured in seconds where $t \geq 0$.

a. Find the maximum speed of the aeroplane. Give your answer in ms^{-1}. 3 marks

$\underset{\sim}{v}(t) = \left(-150\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{i} + \left(200\frac{\pi}{6}\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j}$

$\underset{\sim}{v}(t) = \left(-25\pi\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{i} + \left(\frac{100\pi}{3}\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j}$

Speed $= |\underset{\sim}{v}| = \sqrt{\left(-25\pi\cos\left(\frac{\pi t}{6}\right)\right)^2 + \left(\frac{100\pi}{3}\sin\left(\frac{\pi t}{6}\right)\right)^2}$

$= \sqrt{625\pi^2\cos^2\left(\frac{\pi t}{6}\right) + \frac{10000\pi^2}{9}\sin^2\left(\frac{\pi t}{6}\right)}$

$= \frac{100}{3}\pi\sqrt{\frac{9}{16}\cos^2\left(\frac{\pi t}{6}\right) + \sin^2\left(\frac{\pi t}{6}\right)}$ need to take this as a factor

$625 \div \frac{10000}{9} = \frac{9}{16}$.

$= \frac{100}{3}\pi\sqrt{\frac{9}{16}\cos^2\left(\frac{\pi t}{6}\right) + 1 - \cos^2\left(\frac{\pi t}{6}\right)}$

$\sin^2 = 1 - \cos^2$.

$= \frac{100}{3}\pi\sqrt{1 - \frac{7}{16}\cos^2\left(\frac{\pi t}{6}\right)}$

Max when $\cos\left(\frac{\pi t}{6}\right) = 0$ Max Speed $= \frac{100}{3}\pi\sqrt{1} = \frac{100}{3}\pi$

b. i. Use $\underset{\sim}{r}_A(t)$ to show that the cartesian equation of the path of the aeroplane is given by

$$\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = 1.$$

2 marks

$x = 450 - 150\sin\left(\frac{\pi t}{6}\right)$ $y = 400 - 200\cos\left(\frac{\pi t}{6}\right)$

$\frac{x-450}{150} = \sin\left(\frac{\pi t}{6}\right)$ $\frac{y-400}{-200} = \cos\left(\frac{\pi t}{6}\right)$

$\left(\frac{x-450}{150}\right)^2 = \sin^2\left(\frac{\pi t}{6}\right)$ $\left(\frac{y-400}{200}\right)^2 = \cos^2\left(\frac{\pi t}{6}\right)$

$\sin^2\left(\frac{\pi t}{6}\right) + \cos^2\left(\frac{\pi t}{6}\right) = 1$

$\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = 1$

Part i equation is an ellipse. Centre (450, 400)

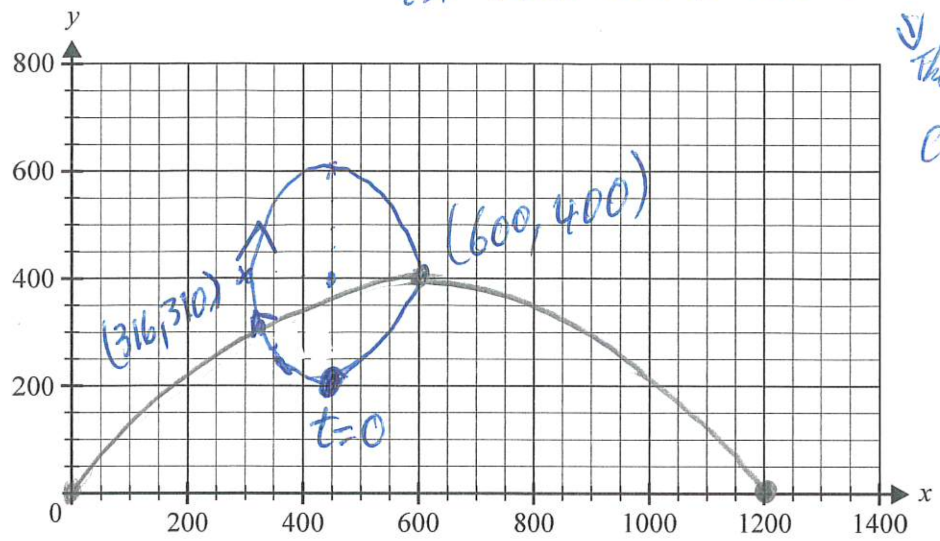
ii. Sketch the path of the aeroplane on the axes provided below. Label the position of the aeroplane when $t = 0$, using coordinates, and use an arrow to show the direction of motion of the aeroplane.

3 marks

$t=0 \quad x=450 \quad y=200$

$t=1 \quad x=450-\cancel{150}\sin\left(\frac{\pi}{8}\right) = 450-75 = 375.$

Thus going Clockwise



Heaps of Calculator use in this Question

A friend of the pilot launches an experimental jet-powered drone to take photographs of the air show. The position of the drone at time t relative to the fixed origin is given by

$\underset{\sim}{r}_D(t) = (30t)\underset{\sim}{i} + \left(-t^2 + 40t\right)\underset{\sim}{j}$, where t is in seconds and $0 \le t \le 40$, $\underset{\sim}{i}$ is a unit vector in the same horizontal direction, $\underset{\sim}{j}$ is a unit vector vertically up, and displacement components are measured in metres.

c. Sketch the path of the drone on the axes provided in **part b.ii.** Using coordinates, label the points where the path of the drone crosses the path of the aeroplane, correct to the nearest metre.

No decimal places.

3 marks

$x = 30t \qquad y = -t^2 + 40t.$

$t = \frac{x}{30} \rightarrow y = -\left(\frac{x}{30}\right)^2 + 40 \times \frac{x}{30}$

$= \frac{-x^2}{900} + \frac{4x}{3}.$

$= x\left(-\frac{x}{900} + \frac{4}{3}\right).$

Points crossing.

(316, 310) (600, 400)

x-int

$0 = x\left(-\frac{x}{900} + \frac{4}{3}\right)$

$x = 0 \qquad -\frac{x}{900} + \frac{4}{3} = 0$

$\frac{4}{\cancel{3}} \times \cancel{900}^{300} = x$

$x = 1200$

Max when $x = 600$

$y = -\frac{(600)^2}{900} + \frac{4 \times 600}{3}$

$= 400$

d. Determine whether the drone will make contact with the aeroplane. Give reasons for your answer. 3 marks

If collide $x_A = x_B$ and $y_A = y_B$ at same time

$450 - 150\sin\left(\frac{\pi t}{6}\right) = 30t$ $\quad 400 - 200\cos\left(\frac{\pi t}{6}\right) = -t^2 + 40t$

$t = 12.85$ $\quad t = 10, 14.73, 21, 26.58$

No time common for x and y.

$\therefore$ Don't collide (make contact)

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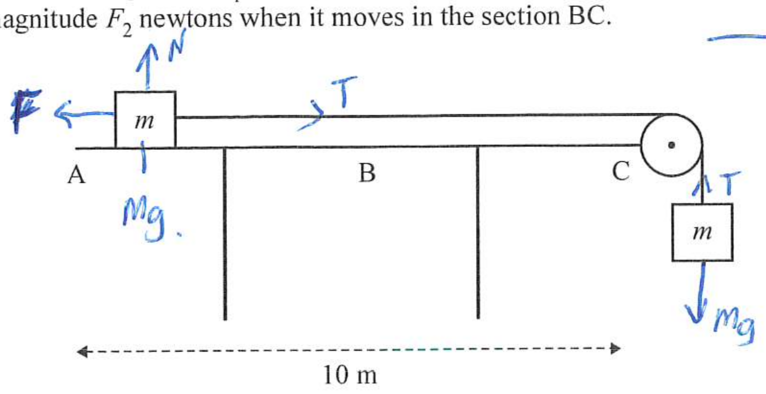
Question 5 (13 marks)

Two objects, each of mass m kilograms, are connected by a light inextensible string that passes over a smooth pulley, as shown below. The object on the platform is initially at point A and, when it is released, it moves towards point C. The distance from point A to point C is 10 m. The platform has a rough surface and, when it moves along the platform, the object experiences a horizontal force opposing the motion of magnitude F_1 newtons in the section AB and a horizontal force opposing the motion of magnitude F_2 newtons when it moves in the section BC.

$\longrightarrow$ +ve.



a. On the diagram above, mark all forces that act on each object once the object on the platform has been released and the system is in motion. 2 marks

The force F_1 is given by $F_1 = kmg$, $k \in R^+$.

b. **i.** Show that an expression for the acceleration, in ms^{-2}, of the object on the platform, in terms of k, as it moves from point A to point B is given by $\frac{g(1-k)}{2}$. 2 marks

$\underset{\sim}{R} = m\underset{\sim}{a}$.

Using the whole System.

$mg - F_1 = (m+m)\underset{\sim}{a}$.

$mg - kmg = 2m\underset{\sim}{a}$

$mg(1-k) = 2m\underset{\sim}{a}$

$\underset{\sim}{a} = \frac{\cancel{m}g(1-k)}{2\cancel{m}} = \frac{g(1-k)}{2}$

ii. The system will only be in motion for certain values of k.

Find these values of k. From information $k \in R^+$ 1 mark

$\underset{\sim}{a}$ must be +ve.

$k < 1$

So not necessary to say $0 < k < 1$

Remember R^+ does not include 0

Point B is midway between points A and C. → distance AB = 5 m.

c. Find, in terms of k, the time taken, in seconds, for the object on the platform to reach point B. 2 marks

$a = \frac{g(1-k)}{2}$ $u = 0$ $x = 5$ $t = ?$ $x = ut + \frac{1}{2}at^2$

$5 = 0 + \frac{1}{2} \times \frac{g(1-k)}{2} \times t^2$

$t^2 = \frac{20}{g(1-k)}$ $t = \sqrt{\frac{20}{g(1-k)}} = 2\sqrt{\frac{5}{g(1-k)}}$

(Variations were accepted).

d. Express, in terms of k, the speed v_B, in ms^{-1}, of the object on the platform when it reaches point B. 2 marks

$v = u + at$

$v_B = 0 + \sqrt{\frac{20}{g(1-k)}} \times \frac{g(1-k)}{2} = \sqrt{5g(1-k)}$

Could use $v^2 = u^2 + 2ax$.

e. When the object on the platform is at point B, the string breaks. The velocity of the object at point B is $v_B = 2.5\ \text{ms}^{-1}$. The force that opposes motion from point B to point C is $F_2 = 0.075\ mg + 0.4\ mv^2$, where v is the velocity of the object when it is a distance of x metres from point B. The object on the platform comes to rest before point C. → Now only force is friction

Find the object's distance from point C when it comes to rest. Give your answer in metres, correct to two decimal places. 4 marks

$v_B = 2.5$ m/s.

Let $x = 0$ at B

$\underset{\sim}{R} = m\underset{\sim}{a}$.

$-(0.075mg + 0.4mv^2) = m\underset{\sim}{a}$.

$a = -0.075g - 0.4v^2$.

$v\frac{dv}{dx} = -0.075g - 0.4v^2$

$\frac{dv}{dx} = \frac{-0.075g - 0.4v^2}{v}$

$\frac{dx}{dv} = \frac{v}{-0.075g - 0.4v^2}$

$x = \int \frac{v}{-0.075g - 0.4v^2}\,dv$

$x = 1.25\log_e|-0.4x^2 - 0.075g| + C$.

But we want to know x from $v = 2.5 \rightarrow v = 0$

Sub. $g = 9.8$

$x = \int_{2.5}^{0} \frac{v}{-0.075g - 0.4v^2}\,dv$

$= 1.85$.

Distance from C.

$= 5 - 1.85$

$= 3.15$ m.