VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2021

ANS.

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER									Letter

MATHEMATICAL METHODS

Written examination 1

Wednesday 3 November 2021

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 11 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (3 marks)

a. Differentiate $y = 2e^{-3x}$ with respect to x. 1 mark

$\frac{dy}{dx} = -6e^{-3x}$

b. Evaluate $f'(4)$, where $f(x) = x\sqrt{2x+1}$. 2 marks

$u = x \quad v = (2x+1)^{\frac{1}{2}}$

$u' = 1 \quad v' = \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2.$

$= \frac{1}{\sqrt{2x+1}}$

$f'(x) = \frac{x}{\sqrt{2x+1}} + \sqrt{2x+1}$

$f'(4) = \frac{4}{\sqrt{2\times4+1}} + \sqrt{2\times4+1}$

$= \frac{4}{\sqrt{9}} + \sqrt{9}.$

$= \frac{4}{3} + 3$

$= \frac{13}{3}$

Question 2 (2 marks)

Let $f'(x) = x^3 + x$.

Find $f(x)$ given that $f(1) = 2$.

$f(x) = \int (x^3 + x)\, dx$

$f(x) = \frac{x^4}{4} + \frac{x^2}{2} + C$

$f(1) = 2.$

$2 = \frac{1^4}{4} + \frac{1^2}{2} + C.$

$2 = \frac{3}{4} + C$

$C = \frac{5}{4}$

$f(x) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4}$

Question 3 (5 marks)

Consider the function $g : R \to R, g(x) = 2\sin(2x)$.

a. State the range of g. 1 mark

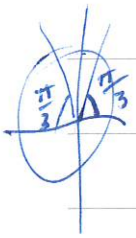
Range g $[-2, 2]$

b. State the period of g. 1 mark

Period $= \frac{2\pi}{2} = \pi$

c. Solve $2\sin(2x) = \sqrt{3}$ for $x \in R$. 3 marks

$\sin(2x) = \frac{\sqrt{3}}{2}$.



$2x = 2n\pi + \frac{\pi}{3}$, $(2n+1)\pi - \frac{\pi}{3}$.

$x = \frac{2n\pi}{2} + \frac{\pi}{6}$, $\frac{(2n+1)}{2}\pi - \frac{\pi}{6}$

$= n\pi + \frac{\pi}{6}$. , $(n + \frac{1}{2})\pi - \frac{\pi}{6}$. $\frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$.

$= (n + \frac{1}{6})\pi$, $n\pi + \frac{1}{2}\pi - \frac{\pi}{6}$.

$n\pi + \frac{1}{3}\pi$

$(n + \frac{1}{3})\pi$. , $n \in Z$

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Question 4 (4 marks)

a. Sketch the graph of $y = 1 - \frac{2}{x-2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates. 3 marks

y-intercept $x=0$

$y = 1 - \frac{2}{0-2}$

$= 1 + 1$

$= 2.$

x-intercept $y=0$

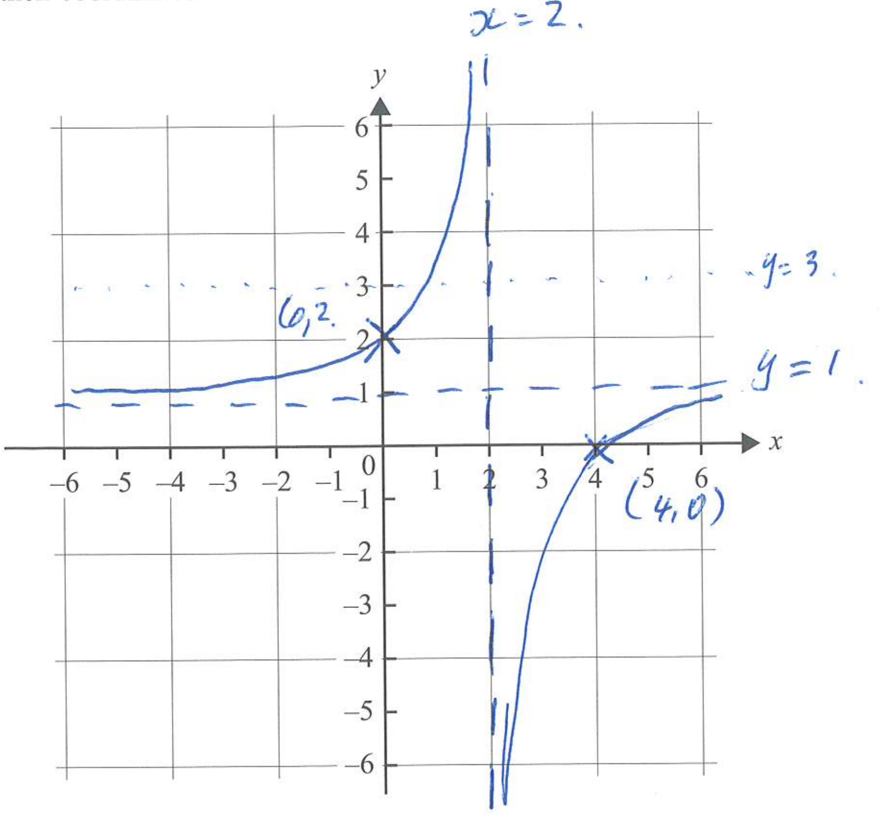
$0 = 1 - \frac{2}{x-2}$

$\frac{2}{x-2} = 1$

$\frac{x-2}{2} = 1$

$x-2 = 2.$

$x = 4.$



b. Find the values of x for which $1 - \frac{2}{x-2} \geq 3$. 1 mark

$1 - \frac{2}{x-2} = 3.$

$-\frac{2}{x-2} = 2.$

$-2 = 2(x-2)$

$-1 = x-2$

$1 = x.$

$x \in [1, 2)$

Question 5 (4 marks)

Let $f : R \to R$, $f(x) = x^2 - 4$ and $g : R \to R$, $g(x) = 4(x - 1)^2 - 4$.

a. The graphs of f and g have a common horizontal axis intercept at $(2, 0)$.

Find the coordinates of the other horizontal axis intercept of the graph of g. 2 marks

$0 = 4(x-1)^2 - 4$

$4 = 4(x-1)^2$

$1 = (x-1)^2$

$\pm 1 = x - 1$

$x = 1 \pm 1$

$x = 1 + 1$ $\quad$ $x = 1 - 1$

$= 2$ $\quad$ $= 0$.

other intercept $(0, 0)$.

b. Let the graph of h be a transformation of the graph of f where the transformations have been applied in the following order:

- dilation by a factor of $\frac{1}{2}$ from the vertical axis (parallel to the horizontal axis)
- translation by two units to the right (in the direction of the positive horizontal axis)

State the rule of h and the coordinates of the horizontal axis intercepts of the graph of h. 2 marks

$f(x) = x^2 - 4$.

Dilation $\quad f(2x) = (2x)^2 - 4$.

Translation $\quad f(2(x-2)) = (2(x-2))^2 - 4$

$= 4(x-2)^2 - 4$

$h(x) = 4(x-2)^2 - 4$.

Horizontal axis Intercepts

$0 = 4(x-2)^2 - 4$

$4 = 4(x-2)^2$

$1 = (x-2)^2$

$\pm 1 = x - 2$

$x = 2 \pm 1$

$x = 2+1$ $\quad$ $x = 2-1$

$= 3$ $\quad$ $= 1$

Intercepts $(3, 0)$ $(1, 0)$.

Question 6 (6 marks)

An online shopping site sells boxes of doughnuts.

A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

- glazed, with custard
- glazed, with no custard
- not glazed, with custard
- not glazed, with no custard.

It is known that, in the box:

- $\frac{1}{2}$ of the doughnuts are with custard
- $\frac{7}{10}$ of the doughnuts are not glazed
- $\frac{1}{10}$ of the doughnuts are glazed, with custard.

	Custard	C'	
Glaze	$\frac{1}{10}$	$\frac{2}{10} = \frac{1}{5}$	$\frac{3}{10}$
G'	$\frac{4}{10} = \frac{2}{5}$	$\frac{3}{10}$	$\frac{7}{10}$
	$\frac{1}{2} = \frac{5}{10}$	$\frac{1}{2}$	1

a. A doughnut is chosen at random from the box.

Find the probability that it is not glazed, with custard. 1 mark

$Pr(G' \cap C) = \frac{2}{5}$.

b. The 20 doughnuts in the box are randomly allocated to two new boxes, Box A and Box B.

Each new box contains 10 doughnuts.

One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random.

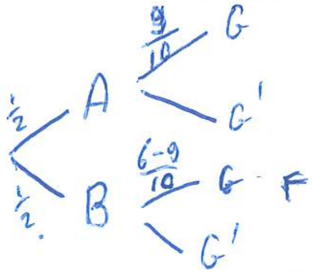
Let g be the number of glazed doughnuts in Box A.

Find the probability, in terms of g, that the doughnut comes from Box B given that it is glazed. 2 marks

$$Pr(B \mid \text{Glazed}) = \frac{Pr(B \cap \text{Glazed})}{Pr(\text{Glazed})} = \frac{\frac{1}{2} \times \frac{6-g}{10}}{\frac{6}{20}}$$

$$= \frac{6-g}{20} \times \frac{20}{6}$$

$$= \frac{6-g}{6}.$$



No. Glazed in the 20 $= \frac{3}{10} \times 20 = 6$.

Glazed in Box $A = g \to$ in Box $B = 6 - g$

$Pr(\text{Glazed}) = \frac{\text{No. Glazed}}{20} = \frac{6}{20}$

Question 6 – continued

c. The online shopping site has over one million visitors per day.
It is known that half of these visitors are less than 25 years old. $\rightarrow p = \frac{1}{2}.$
Let $\hat{P}$ be the random variable representing the proportion of visitors who are less than 25 years old in a random sample of five visitors. $\rightarrow n = 5$
Find $\Pr\left(\hat{P} \geq 0.8\right)$. Do not use a normal approximation. $\rightarrow$ Binomial — 3 marks

$n = 5$

$$\Pr(\hat{p} \geq 0.8) = \Pr\left(\frac{X}{5} \geq 0.8\right)$$
$$= \Pr(X \geq 4).$$
$$= \Pr(X=4) + \Pr(X=5).$$
$$= {}^5C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right) + {}^5C_5\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^0$$
$$= 5 \times \frac{1}{32} + 1 \times \frac{1}{32}$$
$$= \frac{6}{32}$$
$$= \frac{3}{16}.$$

Question 7 (3 marks)

A random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{k}{x^2} & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a positive real number.

a. Show that $k = 2$. 1 mark

$\int_1^2 \frac{k}{x^2}\,dx = 1$

$k\int_1^2 x^{-2}\,dx = 1$

$k\left[\frac{x^{-1}}{-1}\right]_1^2 = 1$

$k\left[\frac{1}{-1\times 2} - \frac{1}{-1\times 1}\right] = 1$

$k\left[-\frac{1}{2} + 1\right] = 1$

$\frac{1}{2}k = 1$

$k = 2.$

b. Find $E(X)$. 2 marks

$E(x) = \int_1^2 x \times \frac{2}{x^2}\,dx$

$= \int_1^2 \frac{2}{x}\,dx.$

$= \left[2\log_e(x)\right]_1^2$

$= 2\log_e(2) - 2\log_e(1).$

$= 2\log_e(2).$

Question 8 (5 marks)

The gradient of a function is given by $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{3}{2}$.

The graph of the function has a single stationary point at $\left(3, \frac{29}{4}\right)$.

a. Find the rule of the function. 3 marks

$y = \int \left(\sqrt{x+6} - \frac{x}{2} - \frac{3}{2}\right) dx$

$y = \int \left((x+6)^{\frac{1}{2}} - \frac{x}{2} - \frac{3}{2}\right) dx$

$y = \frac{2(x+6)^{\frac{3}{2}}}{3} - \frac{x^2}{4} - \frac{3}{2}x + C.$

Using $\left(3, \frac{29}{4}\right)$.

$\frac{29}{4} = \frac{2(3+6)^{\frac{3}{2}}}{3} - \frac{3^2}{4} - \frac{3}{2} \times 3 + C.$

$\frac{29}{4} = \frac{2(9)^{\frac{3}{2}3}}{3} - \frac{9}{4} - \frac{9}{2} + C.$

$\frac{29}{4} = \frac{54}{3} - \frac{9}{4} - \frac{9}{2} + C.$

$\frac{29}{4} - \frac{54}{3} + \frac{9}{4} + \frac{9}{2} = C.$

$\frac{87 - 216 + 27 + 54}{12} = C.$

$\frac{168 - 216}{12} = C.$

$-\frac{48}{12} = C.$

$C = -4$

$y = \frac{2(x+6)^{\frac{3}{2}}}{3} - \frac{x^2}{4} - \frac{3}{2}x - 4.$

b. Determine the nature of the stationary point. 2 marks

When $x = 2$.

$\frac{dy}{dx} = \sqrt{2+6} - \frac{2}{2} - \frac{3}{2}$

$= \sqrt{8} - \frac{5}{2}.$

$= 2\sqrt{2} - \frac{5}{2}$

$\Rightarrow$ Positive gradient

When $x = 3$

$\frac{dy}{dx} = 0$

When $x = 4$

$\frac{dy}{dx} = \sqrt{4+6} - \frac{4}{2} - \frac{3}{2}$

$= \sqrt{10} - \frac{7}{2}$

$\Rightarrow$ Negative

LOCAL MAXIMUM.

$\sqrt{10}$ between $\sqrt{9}$ and $\sqrt{16}$.
Close to $\sqrt{9} \rightarrow 3 \cdot ?$
$\left(\frac{7}{2}\right)^2 = \frac{49}{4} = 12\frac{1}{4}.$ 3

$\sqrt{2} \approx 1 \cdot 414 \rightarrow 2\sqrt{2} \approx 2 \cdot 828$

$\sqrt{2} \approx$ I Wish I knew the Square root of two
1 . 4 1 4 3 6 4 2 3.

Rule of Thumb for approximation of $\sqrt{2}$.

TURN OVER

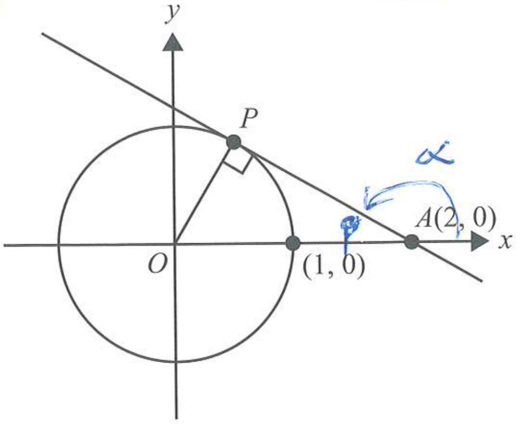
Question 9 (8 marks)

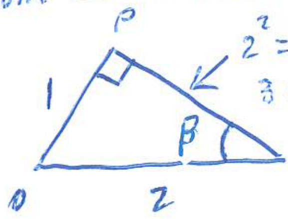
Consider the unit circle $x^2 + y^2 = 1$ and the tangent to the circle at the point P, shown in the diagram below.

radius $= 1$

Geometrically.

From $\triangle$ OPA





$2^2 = 1^2 + PA^2$

$3 = PA^2 \rightarrow PA = \sqrt{3}$

$\sin\beta = \frac{1}{2}$.

$\beta = \frac{\pi}{6}$.

a. Show that the equation of the line that passes through the points A and P is given by $y = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$. 2 marks

$m = \tan\alpha$. $y = mx + c$

$= \tan\left(\frac{5\pi}{6}\right)$ $y = -\frac{1}{\sqrt{3}}x + c$.

$= -\frac{1}{\sqrt{3}}$ $(2, 0)$

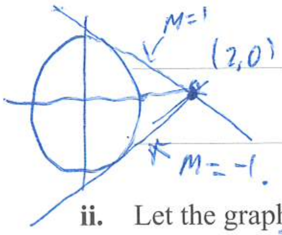
$0 = -\frac{1}{\sqrt{3}} \times 2 + c$

$c = \frac{2}{\sqrt{3}}$.

$y = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$

Let $T : R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, where $q \in R\backslash\{0\}$, and let the graph of the function h be the transformation of the line that passes through the points A and P under T.

b. **i.** Find the values of q for which the graph of h intersects with the unit circle at least once. 1 mark



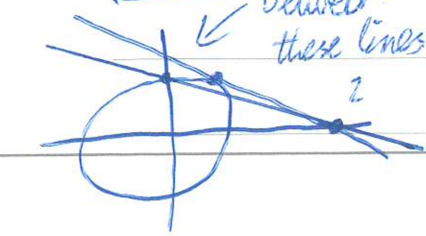
$q \rightarrow$ dilation from x-axis, Could be reflected also

$q \in [-1, 0) \cup (0, 1]$

ii. Let the graph of h intersect the unit circle twice.

~ the line

Find the values of q for which the coordinates of the points of intersection have only positive values. 1 mark



Between these lines

y-coordinate $\in \left(1, \frac{2}{\sqrt{3}}\right)$

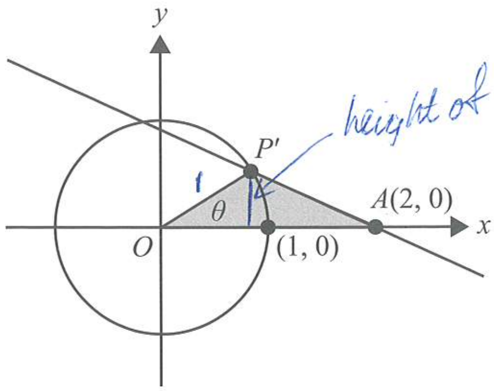
$1 = q \times \frac{2}{\sqrt{3}}$ $\frac{2}{\sqrt{3}} = q \times \frac{2}{\sqrt{3}}$

$q = \frac{\sqrt{3}}{2}$ $q = 1$

$q \in \left(1, \frac{\sqrt{3}}{2}\right)$

c. For $0 < q \leq 1$, let P' be the point of intersection of the graph of h with the unit circle, where P' is always the point of intersection that is closest to A, as shown in the diagram below.

Just means the intersection Point of line and circle in the 1^{st} quadrant.



height of $\triangle = 1 \sin\theta = \sin\theta$

Let g be the function that gives the area of triangle OAP' in terms of θ.

i. Define the function g. 2 marks

Area $= \frac{1}{2} \times$ base $\times$ height

Area $= \frac{1}{2} \times 2 \times \sin\theta$ from Part a.

Area $= \sin\theta$, $\theta \in \left(0, \frac{\pi}{3}\right]$ Also $0 < q \leq 1$

ii. Determine the maximum possible area of the triangle OAP'. 2 marks

Maximum area when θ is a Maximum

ie. When $\theta = \frac{\pi}{3}$.

Maximum Area $= \sin\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2}$ square Units

END OF QUESTION AND ANSWER BOOK