VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2022

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

ANS.

STUDENT NUMBER | | | | | | | | | Letter | |

MATHEMATICAL METHODS

Written examination 1

Friday 27 May 2022

Reading time: 2.00 pm to 2.15 pm (15 minutes)
Writing time: 2.15 pm to 3.15 pm (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 11 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

DO NOT WRITE IN THIS AREA

Question 1 (3 marks)

a. If $y = \sin(x^2 + 1)$, find $\frac{dy}{dx}$. 1 mark

$\frac{dy}{dx} = \cos(x^2+1) \times 2x$

$= 2x \cos(x^2+1)$.

b. If $f(x) = \overset{u}{x^2} \overset{v}{\log_e}(x)$, find $f'(e)$. 2 marks

$u = x^2 \quad v = \log_e(x)$

$u' = 2x \quad v' = \frac{1}{x}$

$f'(x) = x^2 \times \frac{1}{x} + 2x\log_e(x)$

$= x + 2x\log_e(x)$

$f'(e) = e + 2e\log_e(e)$

$= e + 2e$

$= 3e$.

Question 2 (2 marks)

Find $f(x)$, given that $f(0) = 3$ and $f'(x) = \dfrac{2}{x+1} + 2\cos(x)$, where $x > -1$.

$f(x) = \int \left[\frac{2}{x+1} + 2\cos(x)\right] dx$

$= 2\log_e(x+1) + 2\sin(x) + C$.

$f(0) = 3. = 2\log_e(1) + 2\sin(0) + c$

$3 = 0 + 0 + c$

$c = 3$

$f(x) = 2\log_e(x+1) + 2\sin(x) + 3$.

Note: Need $+c$
Base e written as sub-script $\log_e$ Not $\log e$

Question 3 (2 marks) Formula Sheet $\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

A marketing company wants to estimate the proportion of a population who regularly ride bicycles for exercise. The company randomly samples 100 people from this population and finds that 10 of these people regularly ride bicycles for exercise.

Using $z = 2$, find an approximate 95% confidence interval for the true proportion of the population who regularly ride bicycles for exercise.

$n = 100 \quad \hat{p} = \frac{10}{100} = 0.1$

$\left(0.1 - 2\sqrt{\frac{0.1 \times 0.9}{100}}, 0.1 + 2\sqrt{\frac{0.1 \times 0.9}{100}}\right)$

$\left(0.1 - 2\sqrt{\frac{0.09}{100}}, 0.1 + 2\sqrt{\frac{9}{10\,000}}\right)$

$\left(0.1 - 2 \times \frac{3}{100}, 0.1 + 2 \times \frac{3}{100}\right)$

$\left(0.1 - \frac{6}{100}, 0.1 + 0.06\right)$

$(0.04, 0.16)$.

Question 4 (7 marks)

Consider the function $f: [0, 2\pi] \to R, f(x) = a\sin(x) + b$, given that $f\left(\frac{\pi}{2}\right) = 2$ and that $f\left(\frac{3\pi}{2}\right) = -8$.

a. Show that $a = 5$ and $b = -3$. 1 mark

Must have Working out.

$f\left(\frac{\pi}{2}\right) = 2$ $\quad$ $f\left(\frac{3\pi}{2}\right) = -8$

$2 = a\sin\left(\frac{\pi}{2}\right) + b$ $\quad$ $-8 = a\sin\left(\frac{3\pi}{2}\right) + b$

$2 = a + b \quad (1)$ $\quad$ $-8 = -a + b. \quad (2)$

$(1) + 2.$

$-6 = 2b.$

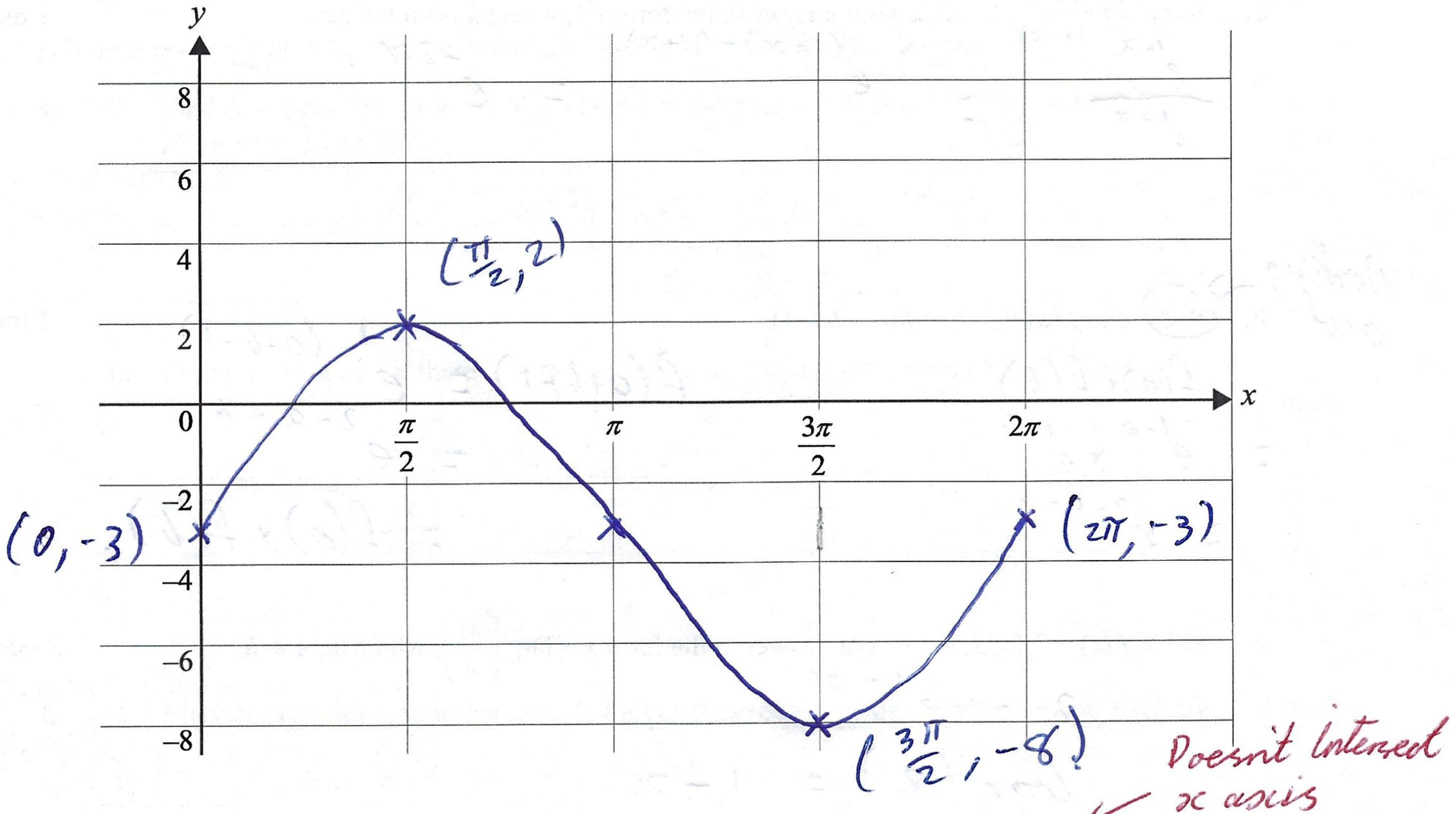
$b = -3.$

Sub into (1)

$2 = a - 3.$

$a = 5.$

b. Sketch the graph of $y = f(x)$ on the set of axes below. Label the endpoints and the turning points with their coordinates. 2 marks



c. State the values of k for which the equation $f(x) + k = 0$, where $k \in R$, has no solution for x. 2 marks

$\{k : k < -2\} \cup \{k : k > 8\}$

d. Find the value of m for which $\int_0^{\pi} (f(x) + m)\,dx = 0$, where $m \in R$. 2 marks

$\int_0^{\pi} (5\sin(x) - 3 + m)\,dx = 0.$

$[-5\cos(x) - 3x + mx]_0^{\pi} = 0.$

$[-5\cos(\pi) - 3\pi + m\pi] - [-5\cos(0) - 3 \times 0 + m \times 0] = 0.$

$[5 - 3\pi + m\pi] - [-5] = 0$

$10 - 3\pi + m\pi = 0$

$m\pi = 3\pi - 10$

$m = \frac{3\pi - 10}{\pi} = 3 - \frac{10}{\pi}.$

DO NOT WRITE IN THIS AREA

Question 5 (4 marks)

Consider the function with rule $f(x) = e^{1-x}$.

a. Simplify $\frac{f(x)}{f(-x)}$. Express your answer in the form e^{kx}, where k is an integer. 1 mark

$\frac{e^{1-x}}{e^{1+x}} = e^{(1-x)-(1+x)} = e^{-2x}$.

Working out

b. Show that $f(a) \times f(b) = f(a+b-1)$. 1 mark

$f(a) \times f(b)$

$= e^{1-a} \times e^{1-b}$

$= e^{2-a-b}$.

$f(a+b-1) = e^{1-(a+b-1)}$

$= e^{2-a-b}$

$= f(a) \times f(b)$.

c. Solve $f(x) = 2$ for x. Give your answer in the form $x = \log_e\left(\frac{m}{n}\right)$, where $m, n \in R$. 2 marks

$2 = e^{1-x}$

$\log_e(2) = 1 - x$

$x = 1 - \log_e(2)$.

$x = \log_e(e) - \log_e(2)$

$x = \log_e\left(\frac{e}{2}\right)$.

	S_M x^2	S'_M $1-x^2$
S_T	$\frac{1}{4}x$	x.
S'_T	$(1-\frac{1}{4}x)$	$(1-x)$.

Question 6 (6 marks)

In a particular city, the probability that it will snow on Monday is x^2.

If it does snow on Monday, the probability that it will snow on Tuesday is $\frac{1}{4}x$.

If it does not snow on Monday, the probability that it will snow on Tuesday is x.

a. **i.** Find an expression, in terms of x, that gives the probability that it will snow on both Monday and Tuesday. 1 mark

$Pr(S_M \text{ and } S_T) = x^2 \times \frac{1}{4}x$

$= \frac{1}{4}x^3$.

Let S_M be Snow Monday

S_T be Snow Tuesday

ii. Find the value of x if there is a 25% chance that it will snow on both Monday and Tuesday. 1 mark

$\frac{25}{100} = \frac{1}{4}x^3$

$\frac{100}{100} = x^3$

$x = 1$

b. **i.** Find an expression, in terms of x, that gives the probability that it will snow on Tuesday. 1 mark

$Pr(S_T) = Pr(S_M \text{ and } S_T) \text{ or } Pr(S'_M \text{ and } S_T)$.

$\frac{1}{4}x^3 + (1-x^2) \times x$

$\frac{1}{4}x^3 + x - x^3 = x - \frac{3}{4}x^3$

ii. Find the value of x that will result in the highest probability that it will snow on Tuesday and find the probability for this value of x in the form $\frac{a}{b}$, where $a, b \in Z$. 3 marks

$\frac{d}{dx}(Pr(S_T)) = 0$

$1 - \frac{9}{4}x^2 = 0$

$1 = \frac{9}{4}x^2$

$\frac{4}{9} = x^2$

$x = \pm\frac{2}{3}$.

$x = \frac{2}{3}$.

$Pr(S_T) = \frac{4}{9}$.

$3\overline{)108}$ 36

$x = 2/3$

$Pr(S_T) = \frac{2}{3} - \frac{3}{4} \times \frac{8}{27}$

$= \frac{2}{3} - \frac{2}{9}$

$= \frac{6-2}{9}$

$= \frac{4}{9}$ ✓

$x = -\frac{2}{3}$

$Pr(S_T) = -\frac{2}{3} - \frac{3}{4} \times \frac{-8}{27}$

$= -\frac{2}{3} + \frac{2}{9}$

$= \frac{-6+2}{9}$

$= -\frac{4}{9}$ X.

DO NOT WRITE IN THIS AREA

Note:
$\rightarrow y = x^4 - x^3 - x^2 + x + 1$
Inverse:
$x = y^4 - y^3 - y^2 + y + 1$
Swap x and y
Thus $p^{-1}(a) = 2$.
already has x and y swapped.

Question 7 (7 marks)

a. Consider the function p, where $p : [1, \infty) \rightarrow R$, $p(x) = x^4 - x^3 - x^2 + x + 1$.

i. Find the value of a when $p^{-1}(a) = 2$, where $a \in R$. 2 marks

$p^{-1}(a) = 2$

$\Rightarrow \quad p(2) = a$

$2^4 - 2^3 - 2^2 + 2 + 1 = a$

$16 - 8 - 4 + 2 + 1 = a$

$a = 7.$

ii. Find the value of b when $p(b) = 1$, where $b > 0$. 2 marks

$b^4 - b^3 - b^2 + b + 1 = 1$

$b^4 - b^3 - b^2 + b = 0$

$b(b^3 - b^2 - b + 1) = 0.$

$b = 0 \qquad b^3 - b^2 - b + 1 = 0$

$\times$

$b > 0 \qquad b = 1$ $\leftarrow$ By inspection

b. Find the rule and the domain of f^{-1}, the inverse of f, if $f : R\backslash\{2\} \rightarrow R$, $f(x) = \dfrac{x+3}{x-2}$. 3 marks

$$\begin{array}{r|l} x-2 & \overline{x+3} \\ & -(x-2) \\ \hline & 5. \end{array}$$

$f(x) = 1 + \dfrac{5}{x-2}.$

Dom $f(x) = R\backslash\{2\}$

Range $f(x) = R\backslash\{1\}$

$x = 1 + \dfrac{5}{y-2}$

$x - 1 = \dfrac{5}{y-2}.$

$\dfrac{x-1}{5} = \dfrac{1}{y-2}.$

$\dfrac{5}{x-1} = y - 2$

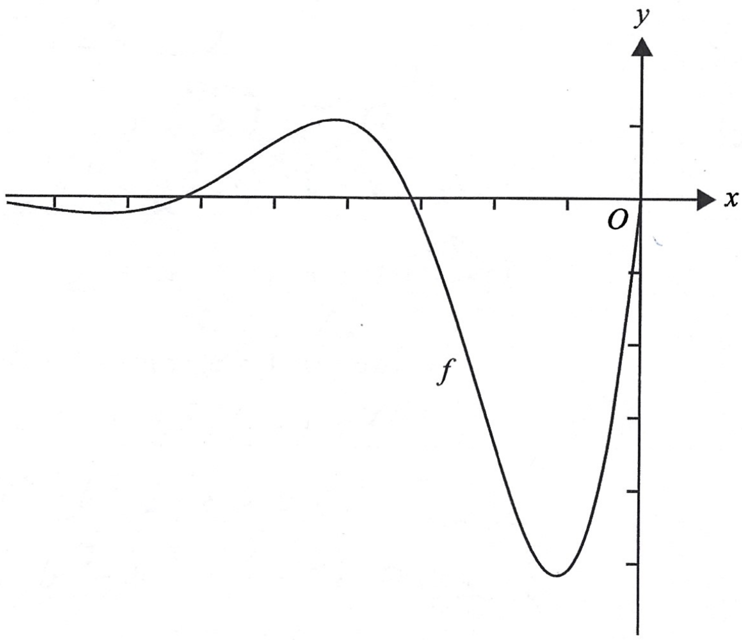
$y = \dfrac{5}{x-1} + 2$

$f^{-1}(x) = \dfrac{5}{x-1} + 2.$

Dom $f^{-1}(x) = R\backslash\{1\}$

Question 8 (9 marks)

Part of the graph of the function f, where $f : (-\infty, 0] \rightarrow R$, $f(x) = e^x \sin(x)$, is shown below.



Note: Domain of x. When defining k

a. Find the general solution to $f(x) = 0$. 2 marks

$e^x \sin(x) = 0$

$e^x = 0$ or $\sin(x) = 0$

No solⁿ

$\sin(x) = 0$.

$x = -k\pi$

where $k \in N \cup \{0\}$

b. Find the general solution to $f'(x) = 0$. 2 marks

$u = e^x \quad v = \sin(x)$

$u' = e^x \quad v' = \cos(x)$

$f'(x) = e^x \cos(x) + e^x \sin x$

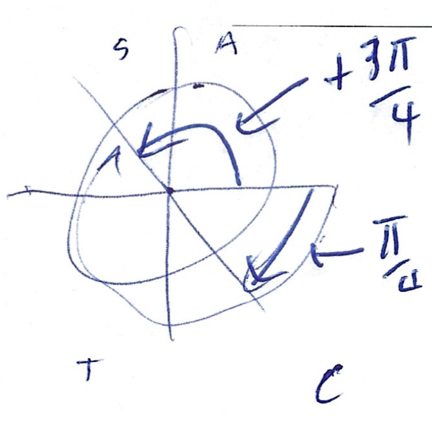
$= e^x(\cos(x) + \sin(x))$

$e^x = 0$

No Solⁿ

$\cos x + \sin x = 0$

$\cos(x) = -\sin(x)$



$x = -(k\pi) - \frac{\pi}{4}$

$= -k\pi + \frac{\pi}{4}$

where $k \in N \cup \{0\}$

Or $\quad x = -(k\pi) + \frac{3\pi}{4}$

c. **i.** Show that $\frac{d}{dx}\left(e^x \sin(x) - e^x \cos(x)\right) = 2e^x \sin(x)$. 1 mark

$e^x \sin(x) + \cancel{e^x \cos(x)} - [\cancel{e^x \cos(x)} - e^x \sin(x)]$

$= e^x \sin(x) + e^x \sin(x)$

$= 2e^x \sin(x).$

ii. Hence, find the area bounded by the graph of the function f and the horizontal axis over the interval $x \in [-\pi, 0]$. 2 marks

$\text{Area} = -\int_{-\pi}^{0} e^x \sin(x)\, dx.$

$= -\frac{1}{2}\int_{-\pi}^{0} 2e^x \sin(x)\, dx$

$= -\frac{1}{2}\left[e^x \sin(x) - e^x \cos(x)\right]_{-\pi}^{0}$

$= -\frac{1}{2}\left[(e^0 \sin(0) - e^0 \cos(0)) - (e^{-\pi} \sin(-\pi) - e^{-\pi} \cos(-\pi))\right]$

$= -\frac{1}{2}\left[(-1) - (e^{-\pi})\right]$

$= \frac{1}{2}\left[e^{-\pi} + 1\right]$

d. Consider the function

$$g: R \to R,\ g(x) = \begin{cases} -\frac{2e^{\pi}}{1+e^{\pi}} e^x \sin(x) & -\pi \le x \le 0 \\ 0 & \text{elsewhere} \end{cases}$$

The continuous random variable X has the probability density function g.

If x_m is the value of x for which g has a maximum value, find $\Pr(X < x_m)$. 2 marks

Max occurs at $x = -\frac{\pi}{4}$.

$\Pr(X < x_m) = \Pr(X < -\frac{\pi}{4})$

$= \int_{-\pi}^{-\frac{\pi}{4}} -\frac{2e^{\pi}}{1+e^{\pi}} e^x \sin(x)\, dx$

Use Info from c

$= -\frac{e^{\pi}}{1+e^{\pi}} \int_{-\pi}^{-\frac{\pi}{4}} 2e^x \sin(x)\, dx.$

$= -\frac{e^{\pi}}{1+e^{\pi}}\left[e^x \sin(x) - e^x \cos(x)\right]_{-\pi}^{-\frac{\pi}{4}}$

$= -\frac{e^{\pi}}{1+e^{\pi}}\left[e^{-\frac{\pi}{4}} \sin(-\frac{\pi}{4}) - e^{-\frac{\pi}{4}} \cos(-\frac{\pi}{4}) - (e^{-\pi} \sin(-\pi) - e^{-\pi} \cos(-\pi))\right]$

$= -\frac{e^{\pi}}{1+e^{\pi}}\left[e^{-\frac{\pi}{4}} \times -\frac{\sqrt{2}}{2} - e^{-\frac{\pi}{4}} \times \frac{\sqrt{2}}{2} - (e^{-\pi})\right]$

$= -\frac{e^{\pi}}{1+e^{\pi}}\left[-\sqrt{2}\, e^{-\frac{\pi}{4}} - e^{-\pi}\right] = \frac{\sqrt{2}\, e^{\frac{3\pi}{4}} + 1}{1+e^{\pi}}$

END OF QUESTION AND ANSWER BOOK

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