VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2022

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

ANS

STUDENT NUMBER | | | | | | | | | Letter | |

MATHEMATICAL METHODS

Written examination 2

Monday 30 May 2022

Reading time: 2.00 pm to 2.15 pm (15 minutes)
Writing time: 2.15 pm to 4.15 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The function f and its inverse, f^{-1}, are one-to-one for all values of x.
If $f(1) = 5$, $f(3) = 7$ and $f(8) = 10$, then $f^{-1}(7)$ and $f^{-1}(5)$ respectively are equal to

3 1

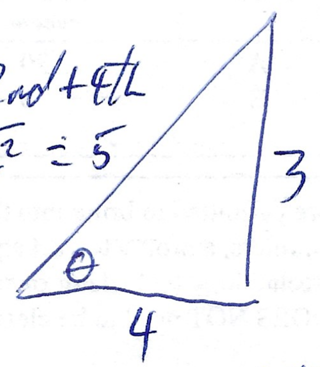
A. 5 and 7
B. 3 and 1
C. 7 and 5
D. 8 and 5
E. 5 and 8

Question 2

If $\tan(\theta) = -\frac{3}{4}$ and $\theta \in [0, 2\pi]$, then $\cos(\theta)$ is equal to

tan -ve in 2nd + 4th

$\sqrt{3^2+4^2} = 5$



$\cos = \frac{4}{5}$.

2nd $\rightarrow -\frac{4}{5}$

4th $\rightarrow \frac{4}{5}$

A. $\frac{3}{5}$ or $-\frac{3}{5}$

B. $\frac{4}{5}$ or $-\frac{3}{5}$

C. $\frac{4}{3}$ or $-\frac{4}{3}$

D. $-\frac{3}{5}$ or $-\frac{4}{5}$

E. $\frac{4}{5}$ or $-\frac{4}{5}$

Question 3

The function f with rule $f(x) = 2\log_e(16 - x)$ has a maximal domain given by

A. $x \in (16, \infty)$
B. $x \in (-\infty, 4)$
C. $x \in (4, \infty)$
D. $x \in (-4, 4)$
E. $x \in (-\infty, 16)$

$16 - x > 0$

$-x > -16$

$x < 16$.

DO NOT WRITE IN THIS AREA

Question 4

Let $g : R \to R$, $g(x) = 3x + a$, where a is a real constant.

Given that $g(g(2)) = 10$, the value of a is

A. -1

B. -2

C. -3

D. -4

E. -5

$g(2) = 3 \times 2 + a$
$= 6 + a$.
$g(g(2)) = 3 \times (6+a) + a$
$10 = 18 + 3a + a$
$10 = 18 + 4a$
$-8 = 4a$
$a = -2$.

Question 5

A continuous random variable, X, has the probability density function, f, given by

$$f(x) = \begin{cases} a \cdot \cos\left(2\pi x + \dfrac{3\pi}{2}\right) & 0 \le x \le \dfrac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

The value of a is

A. $\dfrac{\pi}{4}$

B. $\dfrac{\pi}{2}$

C. π

D. $\dfrac{3\pi}{4}$

E. $\dfrac{3\pi}{2}$

$\int_0^{\frac{1}{2}} a \cos\left(2\pi x + \frac{3\pi}{2}\right) dx = 1$.
$a \int_0^{\frac{1}{2}} \cos\left(2\pi x + \frac{3\pi}{2}\right) dx = 1$
$a \times \frac{1}{\pi} = 1$
$a = \pi$.

Question 6

The line with equation $y = mx + 1$ and the curve with equation $y = 3x^2 + 2x + 4$ intersect at two distinct points.

$\to \Delta > 0$
2 solutions

The values of m are

A. $-4 < m < 8$

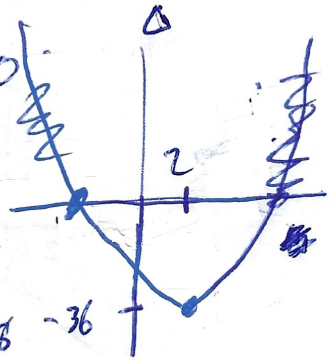
B. $m < -4$

C. ~~$m > 8$~~

D. $m < -4$ or $m > 8$

E. $m = -4$ or $m = 8$

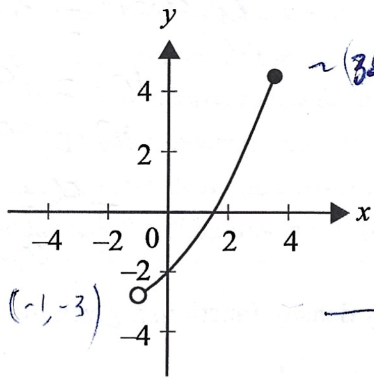
$mx + 1 = 3x^2 + 2x + 4$
$0 = 3x + (2-m)x + 3$.
$\Delta > 0$.
$(2-m)^2 - 4 \times 3 \times 3 > 0$
$(2-m)^2 - 36 > 0$



$2 - m = 6$ $\quad$ $2 - m = -6$
$-m = 4$ $\quad$ $-m = -8$
$m = -4$ $\quad$ $m = 8$
$m < -4$ $\quad$ $m > 8$.

Question 7

The graph of $y = f(x)$ is shown below.



~(3.46, 4.2)

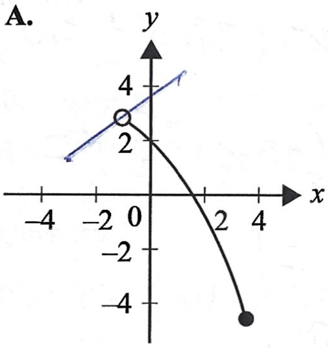
~(−1, −3) → (−3, −1)

$f^{-1}(x)$ → (4.2, 3.8)

Also Needs to curve the other way.

The corresponding graph of the inverse of f, $y = f^{-1}(x)$, is best represented by

A.



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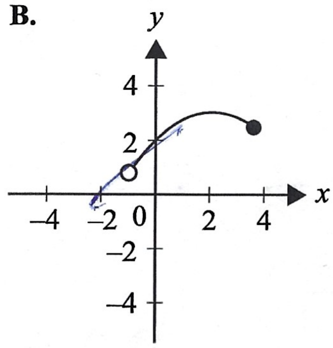

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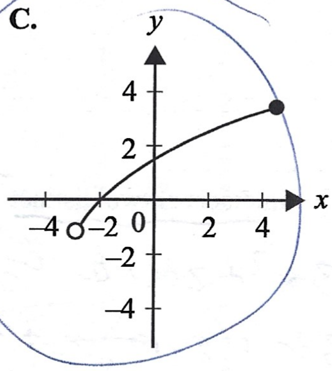

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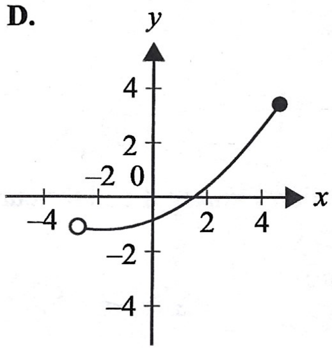
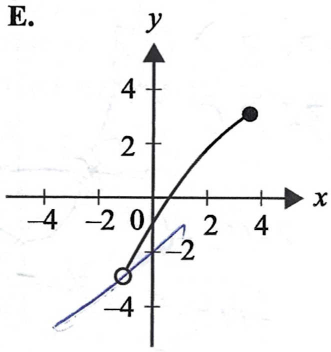


E.



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Question 8

The range of the function with rule $y = \sqrt{4 - x^2} + \log_e(x + 2)$ is contained within the interval

A. $[-4, 2.8]$
B. $(-\infty, 2.8]$
C. $(-4, 2.9)$
D. $(-\infty, 2.9)$
E. $[-4, 2.9)$

Nedc - Graph.
Max at 2.8668
≈ 2.9

Asymptote at $x = -2$.
from $\log_e(x+2)$

Question 9

A survey on sleep habits was conducted using a random sample of 40 people drawn from a large population. Thirty of the survey participants reported that they would like to sleep more.

Using the results of this survey, a 90% confidence interval for the proportion of the population who would like to sleep more would be closest to

A. $\left(0.75 - 1.64\sqrt{\dfrac{0.25 \times 0.75}{40}},\ 0.75 + 1.64\sqrt{\dfrac{0.25 \times 0.75}{40}}\right)$

B. $\left(0.75 - 1.28\sqrt{\dfrac{0.25 \times 0.75}{40}},\ 0.75 + 1.28\sqrt{\dfrac{0.25 \times 0.75}{40}}\right)$

C. $\left(0.75 - 1.96\sqrt{\dfrac{0.25 \times 0.75}{40}},\ 0.75 + 1.96\sqrt{\dfrac{0.25 \times 0.75}{40}}\right)$

D. $\left(30 - 1.28\sqrt{\dfrac{0.25 \times 0.75}{40}},\ 30 + 1.28\sqrt{\dfrac{0.25 \times 0.75}{40}}\right)$

E. $\left(30 - 1.96\sqrt{\dfrac{10 \times 30}{40}},\ 30 + 1.96\sqrt{\dfrac{10 \times 30}{40}}\right)$

$n = 40$

$\hat{p} = \frac{30}{40} = \frac{3}{4} = 0.75$

$z \sim 1.645$

Question 10

The probability distribution for the discrete random variable X is shown in the table below.

x	0	1	2	3
$\Pr(X = x)$	0.1	0.4	0.3	0.2

$\mu = 0 \times 0.1 + 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.2$
$= 0.4 + 0.6 + 0.6$
$= 1.6.$

$\sigma^2 = (0-1.6)^2 \times 0.1 + (1-1.6)^2 \times 0.4 + (2-1.6)^2 \times 0.3 + (3-1.6)^2 \times 0.2$

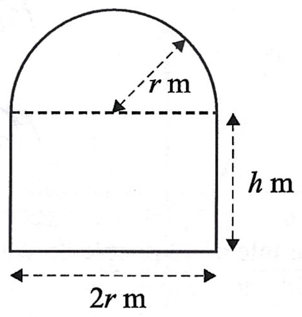
$= 0.84$

The variance of the random variable X is

A. 1.60
B. 1.26
C. 1.00
D. 0.84
E. 0.20

Question 11

The diagram below shows a glass window consisting of a rectangle of height h metres and width $2r$ metres, and a semicircle of radius r metres. The perimeter of the window is 8 m.



$2h+2r+\frac{1}{2}\times 2\pi r = 8$

$2h = 8-2r-\pi r$

$h = 4-r-\frac{\pi}{2}r.$

An expression for the area of the glass window, A, in terms of r is

$A=\frac{1}{2}\pi r^2+(4-r-\frac{\pi}{2}r)\times 2r$

$=\frac{\pi}{2}r^2+8r-2r^2-\pi r^2$

$=8r-2r^2-\frac{\pi}{2}r^2$

A. $A=8r-2r^2-\dfrac{3\pi r^2}{2}$

B. $A=8r-2r^2+\dfrac{\pi r^2}{2}$

C. $A=8r-4r^2-\dfrac{3\pi r^2}{2}$

D. $A=8r-4r^2-\dfrac{\pi r^2}{2}$

E. $A=8r-2r^2-\dfrac{\pi r^2}{2}$

Question 12

The weights of frogs from a certain species have a normal distribution with a mean of 56 g. It is found that 5% of the frogs weigh more than 77 g.

The standard deviation is closest to

A. 3.57

B. 12.77

C. 21.05

D. 163.07

E. 169.11

$\Pr(X>77)=0.05$

$\Pr(Z>z_1)=0.05$

$z_1=1.64485$

$z=\frac{x-\mu}{\sigma}$

$1.64485=\frac{77-56}{\sigma}$

Question 13

Let $f(x)=g(x)\cdot\sqrt{1-x^2}$, where g is a function that is continuous and differentiable for all $x\in R$. $x=0$

The gradient of the tangent to the graph of f at the point where f crosses the vertical axis is equal to

$f'(x)=g'(x)\sqrt{1-x^2}+g(x)\times\frac{-x}{\sqrt{1-x^2}}$

A. 0

B. 1

C. $g(0)$

D. $g'(0)$

E. $g'(0)-g(0)$

$m=g'(0)\sqrt{1-0^2}+g(x)\times\frac{-0}{\sqrt{1-0^2}}$

$=g'(0)+0.$

DO NOT WRITE IN THIS AREA

Question 14

The graph $y = \sin(x)$ is subjected to the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The resulting graph can be described by

A. $y = 1 - \sin(x(x+2))$

B. $y = 1 - \sin(2(x-2))$

C. $y = 1 - \sin\left(\frac{1}{2}(x-2)\right)$

D. $y = 1 - \sin\left(\frac{1}{2}(x+2)\right)$

E. $y = 1 + \sin\left(\frac{1}{2}(x+2)\right)$

$x' = 2x - 2 \rightarrow x = \frac{x+2}{2}$

$y' = -y + 1 \rightarrow y = 1 - y'$

$1 - y' = \sin\left(\frac{x+2}{2}\right)$

$-y' = -1 + \sin\left(\frac{1}{2}(x'+2)\right)$

$y' = 1 - \sin\left(\frac{1}{2}(x'+2)\right)$

Question 15

The probability distribution for the discrete random variable X is shown in the table below.

x	0	1	2	3	4
$\Pr(X = x)$	k	$2k$	$3k$	$4k$	$5k$

When a sample of size $n = 4$ is taken, $\Pr\left(\hat{P} \leq \frac{1}{3}\right)$ is equal to

A. 0

B. 1

C. $\frac{7}{3}$

D. $\frac{1}{15}$

E. $\frac{1}{5}$

$\Pr\left(\frac{X}{n} \leq \frac{1}{3}\right)$

$\Pr\left(\frac{X}{4} \leq \frac{1}{3}\right)$

$\Pr\left(X \leq \frac{4}{3}\right) = k + 2k$

$= 3k$

$k + 2k + 3k + 4k + 5k = 1$

$15k = 1$

$k = \frac{1}{15}$

$= 3 \times \frac{1}{15}$

$= \frac{3}{15}$

$= \frac{1}{5}$

Question 16

Every day, Lucy goes to school by one of three methods: by car, by bus or by walking. The probability that she goes by car is 0.45 and the probability that she goes by bus is 0.2

When Lucy goes by car, the probability that she arrives early is 0.6. When she goes by bus, the probability that she arrives early is 0.1. When she walks, she always arrives early.

What is the probability that Lucy goes to school by car, given that she arrives early?

A. $\frac{27}{64}$

B. $\frac{3}{5}$

C. $\frac{1}{32}$

D. $\frac{9}{34}$

E. $\frac{35}{64}$

$Pr(C|E) = \frac{Pr(C \cap E)}{Pr(E)}$

$Pr(C \cap E) = 0.45 \times 0.6 = 0.27$.

C and E OR B and E OR W and E

$Pr(E) = 0.45 \times 0.6 + 0.2 \times 0.1 + 0.35 \times 1$

$= 0.64$

$Pr(C|E) = \frac{0.27}{0.64} = \frac{27}{64}$

Question 17

The coordinates of the point on a curve with the equation $y = \sqrt{x}$ that are closest to the point (4, 0) are

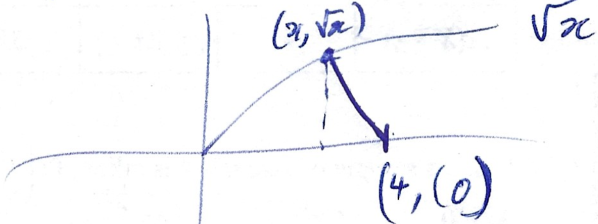
A. (0, 0)

B. $(3, \sqrt{3})$

C. $\left(\frac{7}{2}, \frac{\sqrt{14}}{2}\right)$

D. $\left(\frac{7}{2}, \frac{\sqrt{15}}{2}\right)$

E. (4, 2)



dist $\sqrt{(4-x)^2 + (\sqrt{x})^2}$

dist. $= \sqrt{(4-x)^2 + x}$.

Min dist when $\frac{d}{dx}$(dist) $= 0$

Calc Solve. $x = \frac{7}{2} \rightarrow y = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$

Question 18

At the point where $x = k$, the tangent to the circle given by the equation $x^2 + (y-1)^2 = 1$ meets the positive direction of the x-axis at an angle of 135°.

The value of k could be

A. $-\sqrt{3}$

B. -1

C. $-\frac{1}{\sqrt{2}}$

D. $-\frac{1}{\sqrt{3}}$

E. 0

$m_T = Tan(135°)$

$= -1$

m-Calc.

→ Interactive

→ Calculation

→ impDiff

$\frac{dy}{dx} = \frac{-2x}{2y-2}$

$-1 = \frac{-2x}{2y-2}$.

$x = k$.

$\rightarrow y = \pm\sqrt{1-k^2} + 1$

$-1 = \frac{-2k}{2(\pm\sqrt{1-k^2}+1)-2}$

$k = -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$

Question 19

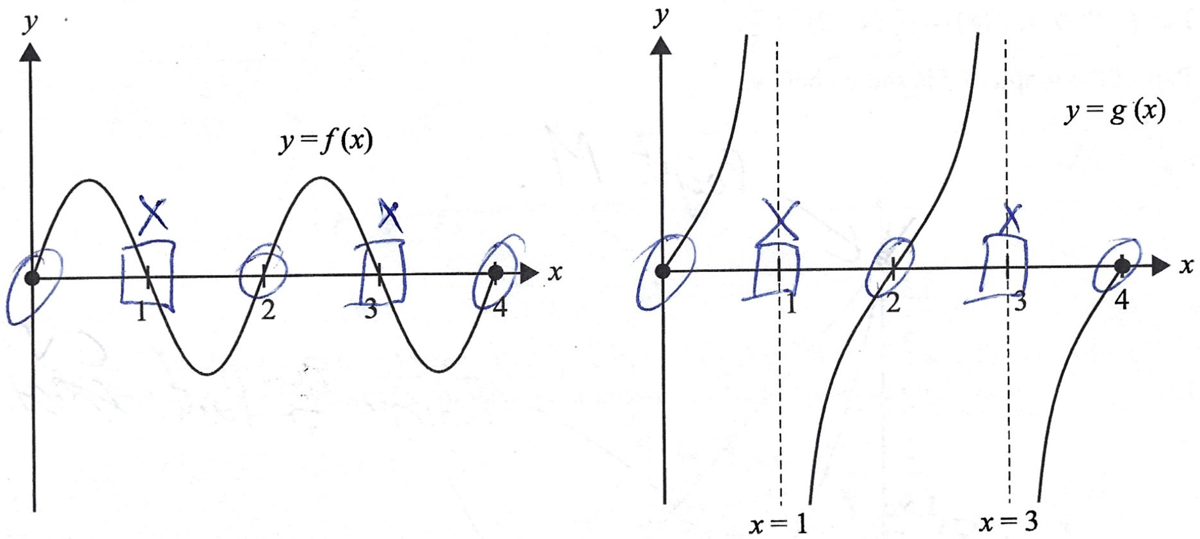
The set of values of p for which $x^3 - px + 2 = 0$ has three distinct, real solutions is

A. $(3, \infty)$
B. $(-\infty, -3)$
C. $(-3, 3)$
D. $(-\infty, 3]$
E. $[3, \infty)$

$x^3 - 5x + 2 \leftarrow 3.$
$x^3 + 5x + 2 \leftarrow 1$
$x^3 - x + 2 \leftarrow 1$
$x^3 - 3x + 2. \leftarrow 2.$

Question 20

Consider the graphs of two circular functions, f and g, shown on the axes below.



On the interval $x \in [0, 4]$, the number of x-intercepts for the graph of the product function $h = f \times g$ is

A. 3
B. 4
C. 5
D. 6
E. 7

$f \times g = 0.$

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

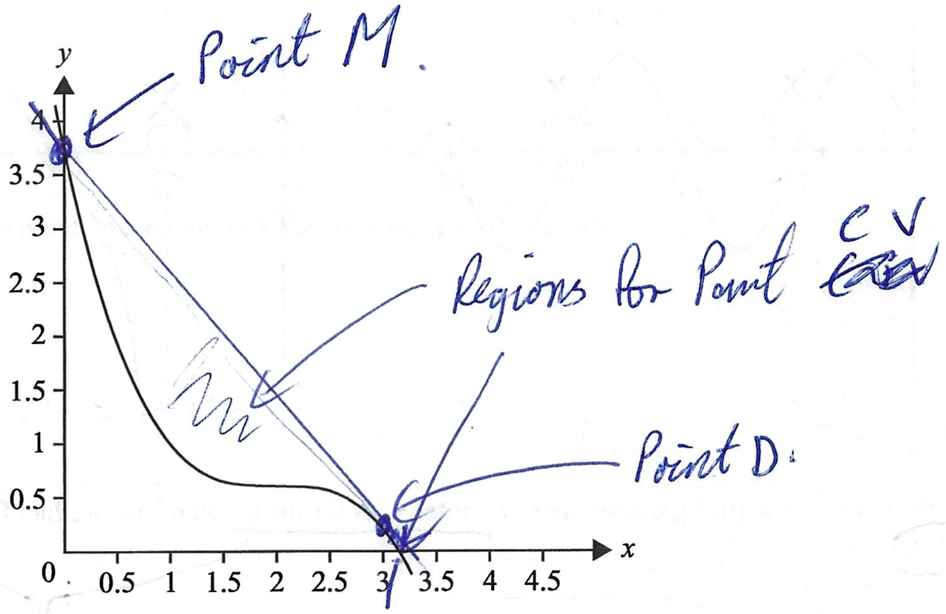
In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (10 marks)

Let $f: R \to R, f(x) = -\frac{2}{5}(x-2)^3 + \frac{3}{5}$.

Part of the graph of f is shown below.



a. Find $f'(x)$, the derivative of f, with respect to x. 1 mark

$f'(x) = -\frac{6}{5}(x-2)^2$

b. Give the coordinates of the stationary point of f. 1 mark

$f'(x) = 0 \to x = 2 \quad f(2) = \frac{3}{5} \qquad (2, \frac{3}{5})$.

c. The graph of f has a tangent with a gradient of $-\frac{6}{5}$ when $x = 1$.

The graph of f also has a tangent with a gradient of $-\frac{6}{5}$ at another point, D.

Working Required

i. Show that the x-coordinate of D is 3. 1 mark

$f'(x) = -\frac{6}{5} = -\frac{6}{5}(x-2)^2$

$(x-2)^2 = 1 \qquad x = 2 \pm 1$

$x - 2 = \pm 1 \qquad x = 3, 1.$

Thus other is $x = 3$.

ii. Determine the equation of the tangent that touches the graph of f at point D. 1 mark

$m = -\frac{6}{5}$ $\quad x = 3 \quad f(3) = \frac{1}{5}$

$y - \frac{1}{5} = -\frac{6}{5}(x-3).$

$y - \frac{1}{5} = -\frac{6}{5}x + \frac{18}{5} \rightarrow y = -\frac{6}{5}x + \frac{19}{5}.$

iii. The tangent to f at point D intersects the graph of f at another point, M.

Give the coordinates of point M. 2 marks

$f(x) = y.$

$-\frac{2}{5}(x-2)^3 + \frac{3}{5} = -\frac{6}{5}x + \frac{19}{5}$

$M \rightarrow x = 0$ $\quad x = 3$ ← Already know this one.

$(0, \frac{19}{5}).$

iv. Find the obtuse angle, in degrees, that the tangent to f at point D makes with the positive direction of the horizontal axis. Give your answer correct to one decimal place. 1 mark

$m = \tan\theta \qquad -\frac{6}{5} = \tan\theta$

$\theta = \tan^{-1}(-\frac{6}{5}) = 129.8^0$

v. The graph has two regions.

The first region is bounded by the graph of f and the tangent to f at point D.

The second region is bounded by the graph of f, the tangent to f at point D and the horizontal axis.

Draw tangent on graph to help Visualise

Find the total area of the two regions. Give your answer correct to four decimal places. 3 marks

$\text{Area} = \int_0^3 [(-\frac{6}{5}x + \frac{19}{5}) - (-\frac{2}{5}(x-2)^3 + \frac{3}{5})]dx$

$+ \int_3^{3.14471} [(-\frac{6}{5}x + \frac{19}{5}) - (-\frac{2}{5}(x-2)^3 + \frac{3}{5})]dx$

$+ \int_{3.14471}^{3.16667} (-\frac{6}{5}x + \frac{19}{5})\,dx.$

To point of intersection with x-axis.

Alternative – Area under Tangent – Area under Curve.

$\text{Area} = \int_0^{3.16667} (-\frac{6}{5}x + \frac{19}{5})dx - \int_0^{3.14471} (-\frac{2}{5}(x-2)^3 + \frac{3}{5})dx$

$= 6.0166667 - 3.315121$

$= 2.7015$ square Units

Question 2 (12 marks)

Sally is using graph sketching software to design the landscape of the four hills shown in Figure 1 below.

She starts by using the square root functions h, h_1 and h_2 to model the shapes of three of the four hills, as shown in Figure 2 below.

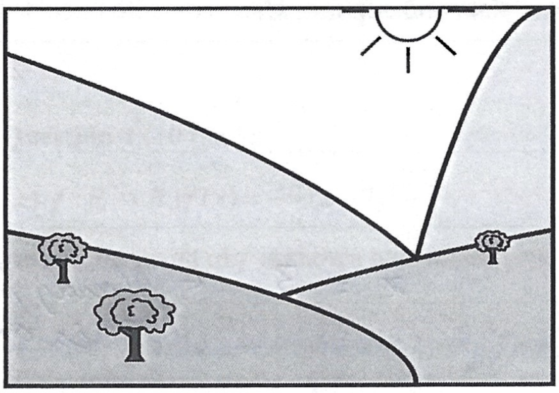


Figure 1

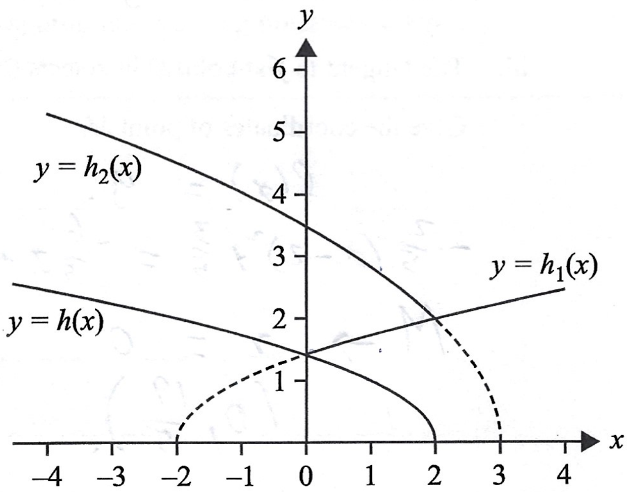


Figure 2

The rule for the function h is $h(x) = \sqrt{2 - x}$.

a. **i.** State the maximal domain for h. 1 mark

$x \in (-\infty, 2]$

ii. The rule for the function h_1 is obtained by reflecting the graph of h in the vertical axis.

State the rule for the function h_1. 1 mark

$h_1(x) = \sqrt{x+2}$

b. The rule for the function h_2 is $h_2(x) = 2\sqrt{3-x}$. There's another possible pair

i. Write a sequence of two transformations that map the graph of h onto the graph of h_2. 1 mark

- Translate 1 unit in the positive x-direction
- Dilation of factor 2 from the x-axis.

ii. Let $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ be a transformation that maps the graph of h onto the graph of h_2.

Find **one** set of possible values for a, b, c and d. 2 marks

$a = 1$ $b = 2$.

$c = 1$ $d = 0$

There is another possible answer related to Alternate Part i ans

iii. Find the value of x for which the slope of the hill defined by the function h is equal to the slope of the hill defined by the function h_2. 1 mark

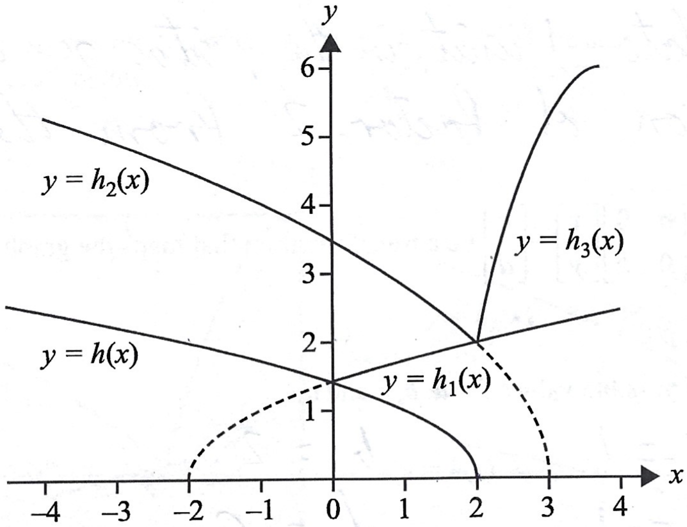
$h'(x) = h_2'(x)$

$\frac{-1}{2\sqrt{2-x}} = \frac{-1}{\sqrt{3-x}}$

$x = \frac{5}{3}$.

DO NOT WRITE IN THIS AREA

Sally decides to use a quadratic function, h_3, to model the shape of the fourth hill in her landscape.



c. Find the rule for h_3, a quadratic function with a stationary point at (4, 6) and which passes through (2, 2). 2 marks

$h_3(x) = a(x-4)^2+6.$

$2 = a(2-4)^2+6.$

$-4 = a(-2)^2$

$-4 = a \times 4$ $\qquad h_3(x) = -(x-4)^2+6$

$a = -1$

Sally believes the function h_3 is closely related to the inverse of h.

d. Find the domain and the rule for the function h^{-1}, the inverse of $h(x) = \sqrt{2-x}$. 2 marks

Range $[0, \infty)$

$h^{-1}(x) = -x^2+2.$

Domain $x \in [0, \infty).$

e. Consider the transformation $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

Does the transformation above map the function h onto the function h_3? Give a reason to justify your answer.

2 marks

$x' = y + 4$ $\quad y' = x + 4$

$y = 4 - x'$ $\quad x = y' - 4.$ $\quad y = \sqrt{2 - x}$

$4 - x' = \sqrt{2 - (y' - 4)}.$

$(4 - x')^2 = 2 - y' + 4$

$(4 - x')^2 = -y' + 6$

$y' = -(\cancel{4} - x')^2 + 6.$

$y' = -(x - 4)^2 + 6.$

Rule is correct, but the domain of this image is $x \in [4, \infty)$.

Thus the transformation cannot give h_3.

Question 3 (14 marks)

Calc - graph.

The functions $p(x) = 2(1 - e^{-x})$ and $q(x) = 2(1 + e^{-x})$ are defined over R.

a. Find the area bounded by the graphs of $p(x)$ and $q(x)$, and by the lines $x = 0$ and $x = 1$. 2 marks

Area $= \int_0^1 2(1+e^{-x})dx - \int_0^1 2(1-e^{-x})dx$

$= -4e^{-1} + 4$ square units

b. State if p and q are each strictly increasing, strictly decreasing or neither. 1 mark

p strictly increasing

q strictly decreasing

c. Find rules for the functions p^{-1} and q^{-1}. 2 marks

$q^{-1}(x) = \log_e\left(\frac{2}{x-2}\right)$ Calc.

$p^{-1}(x) = \log_e\left(\frac{-2}{x-2}\right)$

d. Let point A be the intersection between $p(x)$ and $q^{-1}(x)$, and let point B be the intersection between $q(x)$ and $p^{-1}(x)$.

i. The coordinates of A, correct to three decimal places, are $A(2.329, 1.805)$.

Find the coordinates of B, correct to three decimal places. 1 mark

$(1.805, 2.329)$ Calc.

ii. Write the equation of the line that passes through points A and B in the form $y = mx + c$. Give any approximate values correct to three decimal places. 1 mark

$y = -x + 4.134$ Calc.

Working out Needed.

The function r is the product of functions p and q, with rule given by $r(x) = p(x)q(x)$.

e. **i.** Show that $r(x) = 4\left(1 - e^{-2x}\right)$. 1 mark

$r(x) = 2(1-e^{-x}) \times 2(1+e^{-x})$

$r(x) = 4(1 + \cancel{e^{-x}} - \cancel{e^{-x}} - e^{-2x})$

$r(x) = 4(1 - e^{-2x})$.

ii. State the domain and the range of $r(x)$. 2 marks

Domain R Calc graph.

Range. $(-\infty, 4)$

iii. Find the rule for the function r^{-1}. Give your answer in the form $r^{-1}(x) = \frac{1}{a}\log_e\left(\frac{a^2}{a^2 - x}\right)$. 1 mark

Calc, then put in form required

$r^{-1}(x) = \frac{1}{2}\log_e\left(\frac{-1}{x-4}\right) + \log_e(2)$

$r^{-1}(x) = \frac{1}{2}\left[\log_e\left(\frac{-1}{-1(4-x)}\right) + 2\log_e(2)\right]$

$r^{-1}(x) = \frac{1}{2}\log_e\left(\frac{4}{4-x}\right)$

$= \frac{1}{2}\log_e\left(\frac{2^2}{2^2 - x}\right)$.

Careful Put in Form asked for.

f. **i.** Clearly state the equation of the line that the points of intersection between the function r and its inverse function r^{-1} both lie on. 1 mark

$y = x$.

Note: All functions and inverses intersect on the line $y = x$.

ii. Find the coordinates of the points of intersection between the function r and its inverse function r^{-1}. Give your answers correct to three decimal places. 2 marks

Intersect when $y = x$, thus when $x = 4(1 - e^{-2x})$

$(0, 0)$ $(3.99865, 3.99865)$

$(3.999, 3.999)$.

Note: Could graph $r(x)$ and $r^{-1}(x)$ on calc and find points of intersection.

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Question 4 (15 marks)

A bakery sells glazed doughnuts. The probability density function that describes the thickness of the glaze on a glazed doughnut, X, in millimetres, is given by the function

$$f(x)=\begin{cases}-\frac{1}{108}(x^3-6x^2) & 0\le x\le 6\\ 0 & \text{elsewhere}\end{cases}$$

a. **i.** Show that the mean thickness of the glaze on the bakery's doughnuts, in millimetres, is $\mu=\frac{18}{5}$. 1 mark

$$\mu=\int_0^6 x\times\left(-\frac{1}{108}(x^3-6x^2)\right)dx.$$

$$=\left[\frac{-x^5}{540}+\frac{x^4}{72}\right]_0^6$$

$$=\frac{18}{5}$$

Note: Must be exact answer. millimetres is the unit, careful not to let it distract.

ii. Find the standard deviation of the thickness of the glaze on the bakery's doughnuts, in millimetres. 1 mark

$$\sigma=\sqrt{Var}=\sqrt{\int_0^6\left(x-\frac{18}{5}\right)^2\times-\frac{1}{108}(x^3-6x^2)\,dx}$$

$$=\frac{6}{5}$$

b. **i.** Find the median thickness of the glaze on the bakery's doughnuts, in millimetres, correct to four decimal places. 1 mark

$$\int_0^M -\frac{1}{108}(x^3-6x^2)\,dx=0.5$$

$$M=3.6856.$$

Calc and solve for M. 2 values, but one is outside the domain.

ii. Find the probability that the thickness of the glaze on a randomly selected doughnut is greater than 2 mm, given that the thickness of the glaze is less than the median thickness of the glaze. Give your answer correct to two decimal places. 2 marks

$$\Pr(X>2\mid X<M)=\frac{\Pr(2<X<M)}{\Pr(X<M)}$$

$$\int_2^{3.6856} f(x)\,dx$$

$$=\frac{0.3888788}{0.5}$$

$$=0.77776$$

$$=0.78$$

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The bakery also sells doughnuts with a custard filling. The amount of custard filling in a custard doughnut has a normal distribution with a mean of 22 mL and a standard deviation of 2 mL.

c. Find the probability that a randomly selected custard doughnut has between 21.5 mL and 25 mL of custard filling, correct to four decimal places. 1 mark

$Pr(21.5 < X < 25) = 0.5319$

The bakery also sells doughnuts with a jam filling. The amount of jam filling in a jam doughnut follows a normal distribution. It is known that, on average, 95% of jam doughnuts have at least 15.1 mL of jam filling and 10% of doughnuts have more than 23.9 mL of jam filling.

No decimal places.

d. Find the mean and the standard deviation, correct to the nearest millilitre, of jam filling in jam doughnuts sold by the bakery. 3 marks

Calc. Inverse Norm. and Solve Simultaneous equations

$Pr(X \geq 15.1) = 0.95$ $\quad Pr(X > 23.9) = 0.1$

$z = \frac{x-\mu}{\sigma}$

$Pr(Z \geq z_1) = 0.95$ $\quad Pr(Z > Z_2) = 0.1$

$z_1 = -1.64485$ $\quad Z_2 = 1.281551$

$-1.64485 = \frac{15.1-\mu}{\sigma}$ $\quad = \frac{23.9-\mu}{\sigma}$

$\mu = 20.046$ $\quad \sigma = 3.007$

$\mu = 20$ $\quad \sigma = 3$

A random selection of the bakery's doughnuts is packed into boxes. The proportion of custard doughnuts as a proportion of all doughnuts made in the bakery is 0.44

e. i. Find the expected number of custard doughnuts in a box of 25 doughnuts. 1 mark

$25 \times 0.44 = 11$

ii. Find the probability that in a box of 25 doughnuts there are at most seven custard doughnuts. Use a binomial probability calculation and give your answer correct to four decimal places. 1 mark

$p = 0.44$

$n = 25$

$Pr(X \leq 7) = 0.0773$

$n = ?$ $p = 0.44$

f. Find the minimum number of doughnuts required in a box to ensure that the probability of having at least 12 custard doughnuts in a box is greater than 90%. 1 mark

$Pr(X \geq 12) > 0.9 \rightarrow Pr(X < 12) \leq 0.1$

35 doughnuts.

g. The bakery staff want to determine whether their customers think their doughnuts are delicious.

$\hat{p} = \frac{30}{64} = \frac{15}{32} = 0.46875$

In a random sample of 64 customers, 30 customers said that the bakery's doughnuts were delicious.

Let $\hat{P}$ be the random variable representing the sample proportions in samples of size 64.

i. Find the standard deviation of $\hat{P}$, correct to four decimal places. 1 mark

$sd(\hat{p}) = \sqrt{\frac{\frac{15}{32}(1-\frac{15}{32})}{64}} = 0.0624$

(0.06237781024)

$\leftarrow z = 1.96$

ii. Find an approximate 95% confidence interval using the sample proportion of customers who said that the bakery's doughnuts are delicious, correct to four decimal places. 1 mark

Need to use non rounded z to get accepted answer

$\left(\frac{15}{32} - 1.96 \times 0.06237781024, \frac{15}{32} + 1.96 \times 0.06237781024\right)$

$(0.3465, 0.5910)$

iii. Interpret the confidence interval you found in **part g.ii.** in relation to the proportion of customers who said that the bakery's doughnuts are delicious. 1 mark

For 95% of samples selected, the proportion of those in the sample who think the doughnuts are delicious will be in this range.

Note: for f.

Calc Inv Binomial CDF
Prob - 0.1
Numtried → Trial + Error, until get 12 as the answer.
pos - 0.44

$n = 30 \rightarrow 10$
$n = 40 \rightarrow 14$
$n = 35 \rightarrow 12$.

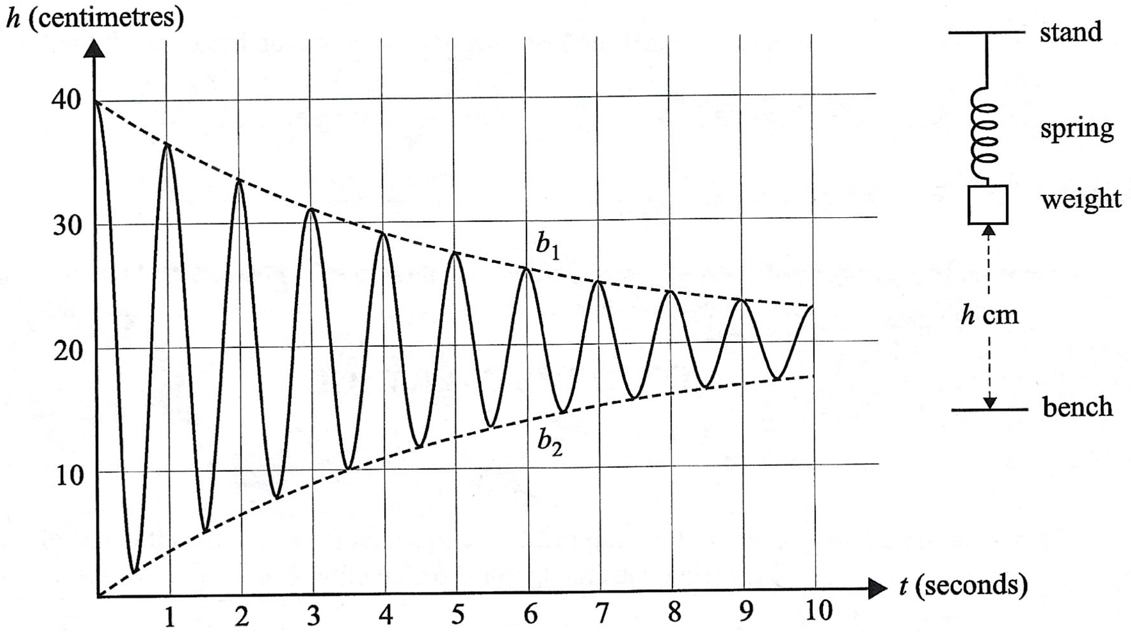
Question 5 (9 marks)

A spring with a weight attached is suspended from a stand. The base of the weight is 40 cm above a bench.

The spring is released and moves vertically up and down above the surface of the bench, such that the height of the base of the weight above the bench over the next 10 seconds is given by the function

$$h(t) = 20e^{-\frac{t}{5}}\cos(2\pi t) + 20, \quad 0 \le t \le 10$$

where t is the time, measured in seconds. A graph of the function h over the first 10 seconds is shown below.



The dashed curve b_1 lies above the graph of h and the dashed curve b_2 lies below the graph of h. Both b_1 and b_2 bound the graph of h.

The dashed curve b_1 has the equation $b_1(t) = 20e^{-\frac{t}{5}} + 20$.

a. State the equation of the dashed curve b_2. 1 mark

$b_2(t) = -20e^{-t/5} + 20.$

b. Find the average value of the height, in centimetres, of the base of the weight above the bench over the first 10 seconds. Give your answer correct to ~~two decimal places.~~ 2 marks

Average Value $= \frac{1}{10-0}\int_0^{10} h(t)\,dt$

$= 20.0087$

$= 20.01$

c. i. Write down the rule for the derivative of h. 1 mark

$h'(t) = -e^{-\frac{t}{5}}\left(4\cos(2\pi t) + 40\pi\sin(2\pi t)\right)$

$= -4e^{-t/5}\left(\cos(2\pi t) + 10\pi\sin(2\pi t)\right)$

ii. Find the time, in seconds, and the height above the surface of the bench, in centimetres, of the point of maximum positive rate of change in h over the first 10 seconds. Write your answer as a coordinate pair, correct to one decimal place. 3 marks

Calc. Graph $h'(t)$ Find Max time

$h'(t) = $ Max. when $t = 0.73987$.

Sub time into h

$h(0.73987) = 18.90286$.

$(0.7, 18.9)$.

d. Determine the total distance travelled by the base of the weight over the first 2 seconds of its motion. Give your answer correct to the nearest centimetre. 2 marks

distance $= (40 - 1.894) + (36.382 - 1.894) + (36.382 - 5.1761)$

$+ (33.4131 - 5.1761) + (33.4131 - 33.4064)$

$= 132.04$

$= 132$ cm

D. Graph $h(t)$ Find Max + Min Points from $0 \leq t \leq 2$ and $h(2)$.

t	0	0.49	0.98	1.49	1.99	2.00
h	40	1.894	36.382	5.1761	33.4131	33.4064
		1	2	3	4	5

END OF QUESTION AND ANSWER BOOK

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