VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2022

SYNNO ANS

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | Letter | |

MATHEMATICAL METHODS
Written examination 1

Wednesday 2 November 2022

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 13 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

DO NOT WRITE IN THIS AREA

Instructions

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (3 marks)

Product Rule.

a. Let $y = 3xe^{2x}$.

Let $u = 3x \quad v = e^{2x}$.

Find $\frac{dy}{dx}$. 1 mark

$\frac{du}{dx} = 3 \quad \frac{dv}{dx} = 2e^{2x}$

$\frac{dy}{dx} = 3x \times 2e^{2x} + 3e^{2x}$

$= 6xe^{2x} + 3e^{2x} = 3e^{2x}(2x+1)$

either accepted

b. Find and simplify the rule of $f'(x)$, where $f : R \to R$, $f(x) = \frac{\cos(x)}{e^x}$. 2 marks

Quotient Rule.

Let $u = \cos(x) \quad v = e^x$

$\frac{du}{dx} = -\sin(x) \quad \frac{dv}{dx} = e^x$

$f'(x) = \frac{e^x \times -\sin(x) - \cos(x)\, e^x}{e^{2x}}$

$= \frac{-e^x(\sin(x) + \cos(x))}{e^{2x}}$

$= \frac{-(\sin(x) + \cos(x))}{e^x}$

Question 2 (4 marks)

a. Let $g:\left(\frac{3}{2},\infty\right)\to R$, $g(x)=\frac{3}{2x-3}$. an antiderivative, thus +C not required

Find the rule for an antiderivative of $g(x)$. 1 mark

Let $u = 2x-3$

$\frac{du}{dx} = 2. \to dx = \frac{1}{2}du$

$\int g(x)\,dx = \int \frac{3}{2x-3}\,dx$

$= 3\int \frac{1}{u}\times\frac{1}{2}du.$

$= \frac{3}{2}\int \frac{1}{u}\,du$

$= \frac{3}{2}\log_e(u)$

$\int g(x)\,dx = \frac{3}{2}\log_e(2x-3)$

b. Evaluate $\int_0^1 \left(f(x)(2f(x)-3)\right)dx$, where $\int_0^1 [f(x)]^2\,dx = \frac{1}{5}$ and $\int_0^1 f(x)\,dx = \frac{1}{3}$. 3 marks

$\int_0^1 (f(x)(2f(x)-3))\,dx$

$= \int_0^1 (2[f(x)]^2 - 3f(x))\,dx$

$= 2\int_0^1 [f(x)]^2\,dx - 3\int_0^1 f(x)\,dx$

$= 2\times\frac{1}{5} - 3\times\frac{1}{3}$

$= \frac{2}{5} - 1$

$= -\frac{3}{5}$

DO NOT WRITE IN THIS AREA

Question 3 (3 marks)

Consider the system of equations

$$kx - 5y = 4 + k$$
$$3x + (k + 8)y = -1$$

Determine the value of k for which the system of equations above has an infinite number of solutions.

Two versions of the same line.
→ Gradient same
→ Y-intercept same.

$kx - 5y = 4 + k.$

$y = \frac{k}{5}x - \left(\frac{4+k}{5}\right)$

$3x + (k+8)y = -1$

$y = -\frac{3}{k+8}x - \left(\frac{1}{k+8}\right)$

Equating gradients

$\frac{k}{5} = -\frac{3}{k+8}$

$k(k+8) = -15$

$k^2 + 8k + 15 = 0$

$(k+3)(k+5) = 0$

$k = -3, -5$

Equating y-intercepts

$\cancel{-}\left(\frac{4+k}{5}\right) = \cancel{-}\left(\frac{1}{k+8}\right)$

$(4+k)(k+8) = 5$

$k^2 + 12k + 32 = 5$

$k^2 + 12k + 27 = 0$

$(k+3)(k+9) = 0$

$k = -3, -9$

For infinite Solutions $k = -3$.

24-05 7005618

Question 4 (5 marks)

A card is drawn from a deck of red and blue cards. After verifying the colour, the card is replaced in the deck. This is performed four times.

Each card has a probability of $\frac{1}{2}$ of being red and a probability of $\frac{1}{2}$ of being blue.

The colour of any drawn card is independent of the colour of any other drawn card.

Let X be a random variable describing the number of blue cards drawn from the deck, in any order.

a. Complete the table below by giving the probability of each outcome. 2 marks

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$.

b. Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red. 1 mark

$\Pr(2R \mid \text{First } B) = \frac{3}{8}$

c. The deck is changed so that the probability of a card being red is $\frac{2}{3}$ and the probability of a card being blue is $\frac{1}{3}$.

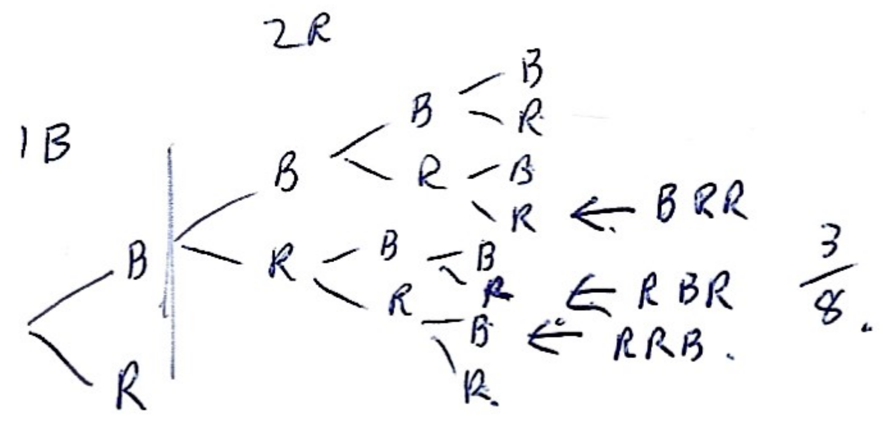
Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red. 2 marks

$\Pr(2R \mid \text{First } B) = \Pr(BRR) + \Pr(RBR) + \Pr(RRB)$.

$= \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}$

$= \cancel{3} \times \frac{4}{\cancel{27}\, 9}$

$= \frac{4}{9}$



Tree Diagram for Part b.

Question 5 (5 marks)

a. Solve $10^{3x-13} = 100$ for x. 2 marks

$10^{3x-13} = 10^2$

$3x - 13 = 2$

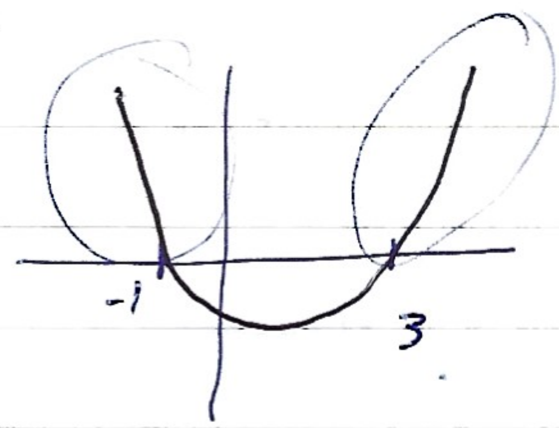
$3x = 15$

$x = 5$

b. Find the maximal domain of f, where $f(x) = \log_e\left(x^2 - 2x - 3\right)$. 3 marks

$x^2 - 2x - 3 > 0$

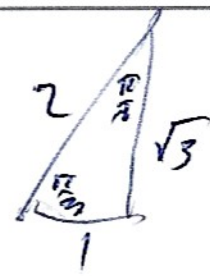
$(x-3)(x+1) > 0$

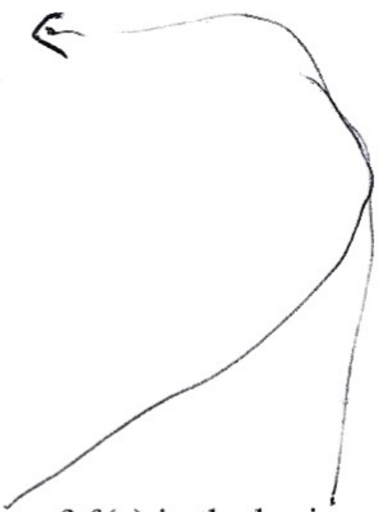


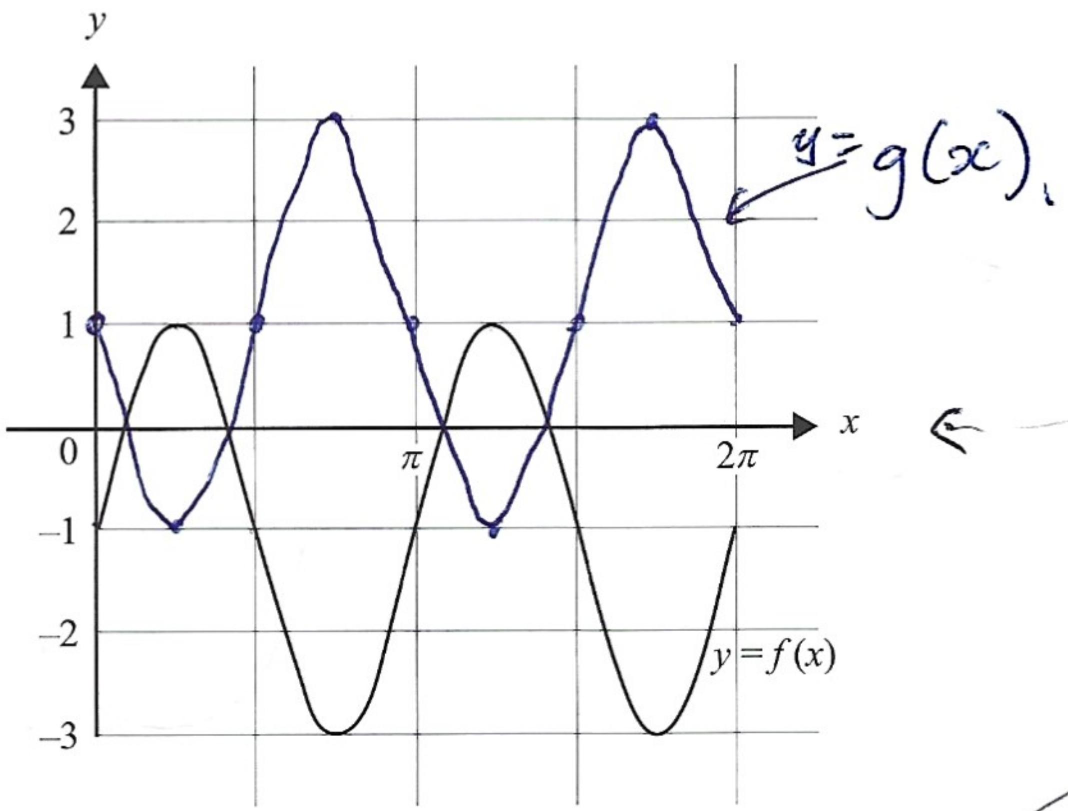
$\text{dom} f = (-\infty, -1) \cup (3, \infty)$

Question 6 (8 marks)

The graph of $y = f(x)$, where $f: [0, 2\pi] \to R$, $f(x) = 2\sin(2x) - 1$, is shown below.







a. On the axes above, draw the graph of $y = g(x)$, where $g(x)$ is the reflection of $f(x)$ in the horizontal axis. 2 marks

b. Find all values of k such that $f(k) = 0$ and $k \in [0, 2\pi]$. 3 marks

$f(k) = 2\sin(2k) - 1 = 0$

$\sin(2k) = \frac{1}{2}$.

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

1st + 2nd Quadrants

$2k = \frac{\pi}{6}, \frac{5\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 4\pi + \frac{\pi}{6}$

$2k = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}$.

$k = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{25\pi}{12}$.

Not in the domain $[0, 2\pi]$

$k = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

c. Let $h : D \to R$, $h(x) = 2\sin(2x) - 1$, where $h(x)$ has the same rule as $f(x)$ with a different domain.
The graph of $y = h(x)$ is translated a units in the positive horizontal direction and b units in the positive vertical direction so that it is mapped onto the graph of $y = g(x)$, where $a, b \in (0, \infty)$.

i. Find the value for b. 1 mark

$b = 2$

Max $f(x) = 1$
Max $g(x) = 3$
$\Rightarrow$ up 2.

ii. Find the smallest positive value for a. 1 mark

$a = \frac{\pi}{2}$.

The point $f(0) = -1$ is translated to $g(\frac{\pi}{2}) = 1$
$\therefore a = \frac{\pi}{2}$.

iii. Hence, or otherwise, state the domain, D, of $h(x)$. 1 mark

Domain $g(x) = [0, 2\pi]$

Thus Domain $h(x) = [-\frac{\pi}{2}, 2\pi - \frac{\pi}{2}]$

$= [-\frac{\pi}{2}, \frac{3\pi}{2}]$

DO NOT WRITE IN THIS AREA

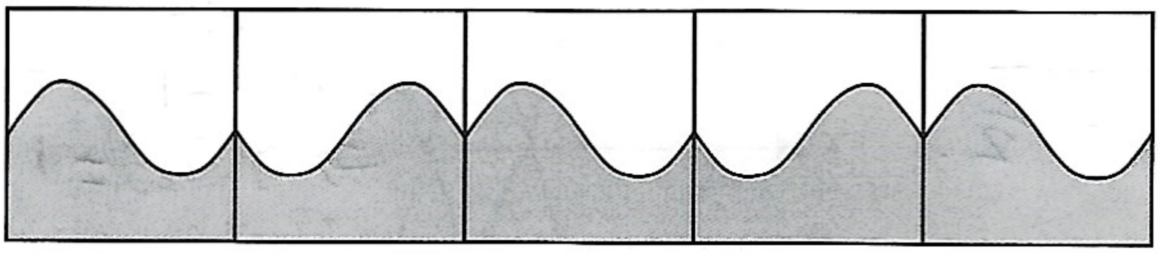
Question 7 (7 marks)

A tilemaker wants to make square tiles of size 20 cm × 20 cm.

The front surface of the tiles is to be painted with two different colours that meet the following conditions:

- Condition 1 – Each colour covers half the front surface of a tile.
- Condition 2 – The tiles can be lined up in a single horizontal row so that the colours form a continuous pattern.

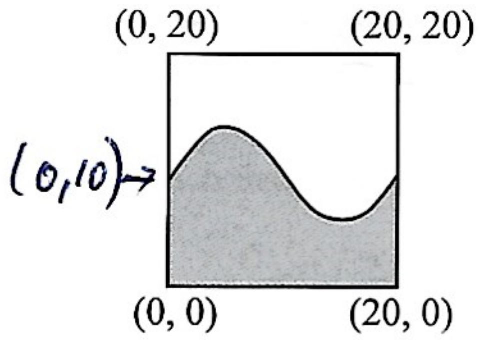
An example is shown below.



There are two types of tiles: Type A and Type B.

For Type A, the colours on the tiles are divided using the rule $f(x) = 4\sin\left(\frac{\pi x}{10}\right) + a$, where $a \in R$.

The corners of each tile have the coordinates (0, 0), (20, 0), (20, 20) and (0, 20), as shown below.



a. **i.** Find the area of the front surface of each tile. 1 mark

Area = 20 × 20

= 400 cm^2.

ii. Find the value of a so that a Type A tile meets Condition 1. 1 mark

$f(0) = 10$

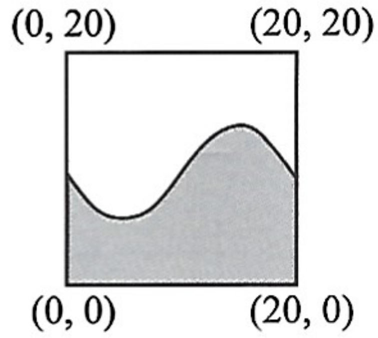
$4\sin(0) + a = 10$

$a = 10$.

Question 7 – continued

Type B tiles, an example of which is shown below, are divided using the rule $g(x) = -\frac{1}{100}x^3 + \frac{3}{10}x^2 - 2x + 10$.

All Working Required.



Half the area $\Rightarrow$ Integral

b. Show that a Type B tile meets Condition 1. 3 marks

$$\text{Area} = \int_0^{20} \left(-\frac{1}{100}x^3 + \frac{3}{10}x^2 - 2x + 10\right) dx.$$

$$= \left[-\frac{x^4}{400} + \frac{3x^3}{30} - \frac{2x^2}{2} + 10x\right]_0^{20}$$

$$= \left[-\frac{x^4}{400} + \frac{x^3}{10} - x^2 + 10x\right]_0^{20}$$

$$= \left[-\frac{20^4}{400} + \frac{20^3}{10} - 20^2 + 10\times 20\right] - \left[-\frac{0^4}{400} + \frac{0^3}{10} - 0^2 + 10\times 0\right]$$

$$= \left[-\frac{160000}{400} + \frac{8000}{10} - 400 + 200\right] - [0]$$

$$= -400 + 800 - 400 + 200$$

$$= 200 \text{ cm}^3 = \tfrac{1}{2} \text{ the area of a tile.}$$

c. Determine the endpoints of $f(x)$ and $g(x)$ on each tile. Hence, use these values to confirm that Type A and Type B tiles can be placed in any order to produce a continuous pattern in order to meet Condition 2. 2 marks

$$f(0) = 4\sin\left(\frac{\pi \times 0}{10}\right) + 10$$
$$= 4\sin(0) + 10$$
$$= 10$$

$$f(20) = 4\sin\left(\frac{20\pi}{10}\right) + 10$$
$$= 4\sin(2\pi) + 10$$
$$= 10$$

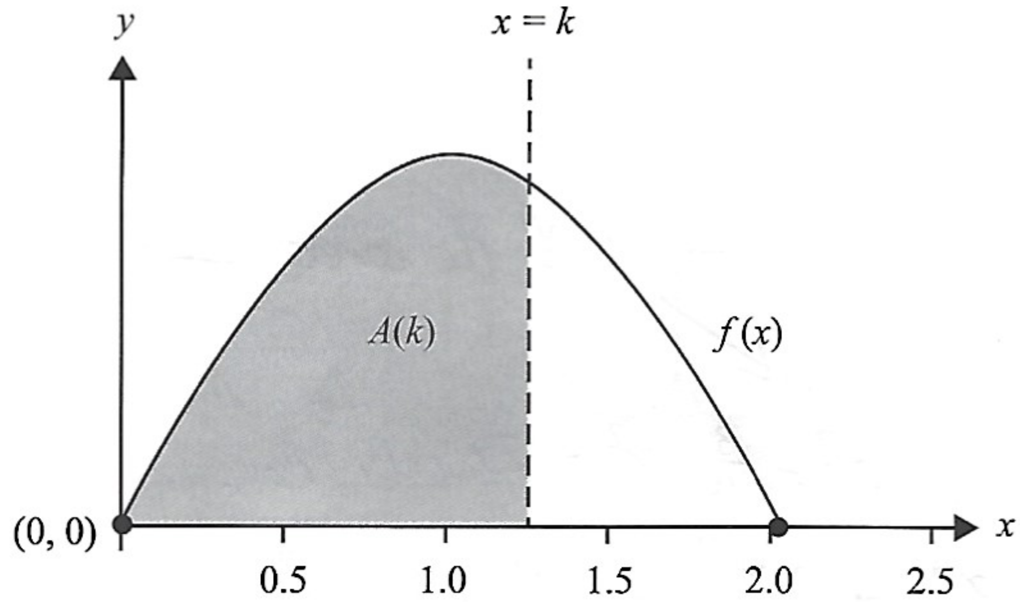
$$g(0) = -\frac{1}{100}0^3 + \frac{3}{10}0^2 - 2\times 0 + 10$$
$$= 10.$$

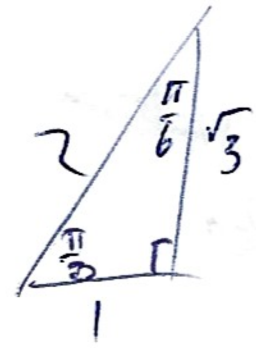
$$g(20) = -\frac{1}{100}\times 20^3 + \frac{3}{10}\times 20^2 - 2\times 20 + 10$$
$$= -\frac{8000}{100} + \frac{1200}{10} - 40 + 10$$
$$= -80 + 120 - 40 + 10$$
$$= 10$$

Since $f(0) = f(20) = g(0) = g(20)$ the tiles can be placed in any order to match.

Question 8 (5 marks)

Part of the graph of $y = f(x)$ is shown below. The rule $A(k) = k\sin(k)$ gives the area bounded by the graph of f, the horizontal axis and the line $x = k$.





a. State the value of $A\left(\frac{\pi}{3}\right)$. 1 mark

$$A\left(\frac{\pi}{3}\right) = \frac{\pi}{3}\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}\pi}{6}.$$

b. Evaluate $f\left(\frac{\pi}{3}\right)$. 2 marks

$$f(x) = A'(k)$$

$$= k\cos(k) + \sin(k).$$

$u = k \quad v = \sin(k)$

$u' = 1 \quad v' = \cos(k)$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3}\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} \times \frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{2}.$$

$$= \frac{\pi + 3\sqrt{3}}{6}$$

Note: Area $= \int f(x)\,dx$. Use Product Rule.

$\Rightarrow f(x) = \frac{d}{dx}$ Area.

Question 8 – continued

c. Consider the average value of the function f over the interval $x \in [0, k]$, where $k \in [0, 2]$.

Find the value of k that results in the maximum average value. 2 marks

$$\text{Average Value} = \frac{A(k)}{k}.$$

$$= \frac{k\sin(k)}{k}$$

$$= \sin(k).$$

Max average value when $\sin(k)$ is Maximum ie. 1.

when $\sin(k) = 1$

$$k = \frac{\pi}{2}.$$

Note:

Could also use Average Value $= \frac{1}{k}\int_0^k f(x)\,dx.$

$$= \frac{1}{k}\left[x\sin(x)\right]_0^k$$

$$= \frac{1}{k}\left[k\sin(k) - 0\sin(0)\right]$$

$$= \frac{1}{k} \times \left[k\sin(k)\right]$$

$$= \sin(k).$$

DO NOT WRITE IN THIS AREA

END OF QUESTION AND ANSWER BOOK