VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

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Victorian Certificate of Education 2022

Letter

STUDENT NUMBER

MATHEMATICAL METHODS

Written examination 2

Thursday 3 November 2022

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 25 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The period of the function $f(x) = 3\cos(2x + \pi)$ is

A. 2π

B. π

C. $\frac{2\pi}{3}$

D. 2

E. 3

$f(x) = 3\cos\left[2\left(x + \frac{\pi}{2}\right)\right]$

$\text{period} = \frac{2\pi}{2} = \pi$

Question 2

The graph of $y = \frac{1}{(x+3)^2} + 4$ has a horizontal asymptote with the equation

A. $y = 4$

B. $y = 3$

C. $y = 0$

D. $x = -2$

E. $x = -3$

Question 3

The gradient of the graph of $y = e^{3x}$ at the point where the graph crosses the vertical axis is equal to

A. 0

B. $\frac{1}{e}$

C. 1

D. e

E. 3

$m = \frac{dy}{dx} = 3e^{3x}$

when $x = 0$

when $x = 0$

$m = 3e^{0}$

$= 3$

Question 4

Which one of the following functions is not continuous over the interval $x \in [0, 5]$?

A. $f(x) = \dfrac{1}{(x+3)^2}$

B. $f(x) = \sqrt{x+3}$

C. $f(x) = x^{\frac{1}{3}}$

D. $f(x) = \tan\left(\dfrac{x}{3}\right)$

E. $f(x) = \sin^2\left(\dfrac{x}{3}\right)$

Use calc. Graph over domain & compare.

Question 5

The largest value of a such that the function $f: (-\infty, a] \to R$, $f(x) = x^2 + 3x - 10$, where f is one-to-one, is

A. -12.25

B. -5

C. -1.5

D. 0

E. 2

Calc find T.P.

T.P. at $(-1.5, -12.5)$

Question 6

Which of the pairs of functions below are **not** inverse functions?

A. $\begin{cases} f(x) = 5x + 3 & x \in R \\ g(x) = \dfrac{x-3}{5} & x \in R \end{cases}$

$x = 5g(x) + 3 \rightarrow g(x) = \frac{x-3}{5}$

B. $\begin{cases} f(x) = \dfrac{2}{3}x + 2 & x \in R \\ g(x) = \dfrac{3}{2}x - 3 & x \in R \end{cases}$

$x = \frac{2}{3}g(x) + 2 \rightarrow g(x) = \frac{3x}{2} - 3.$

C. $\begin{cases} f(x) = x^2 & x < 0 \\ g(x) = \sqrt{x} & x > 0 \end{cases}$

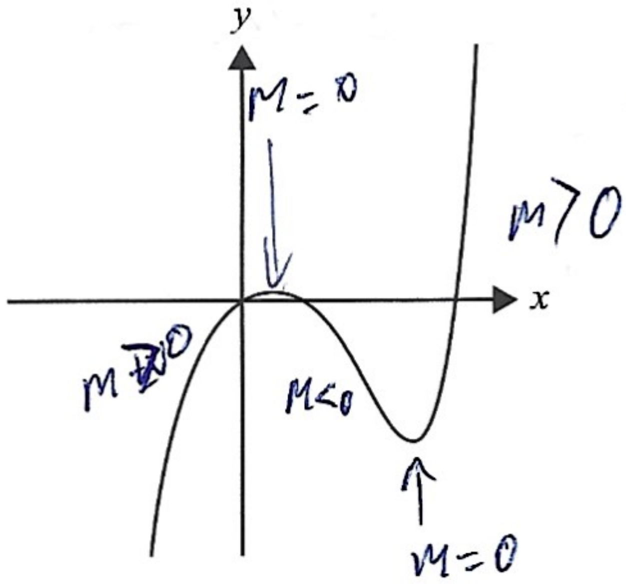
D. $\begin{cases} f(x) = \dfrac{1}{x} & x \neq 0 \\ g(x) = \dfrac{1}{x} & x \neq 0 \end{cases}$

E. $\begin{cases} f(x) = \log_e(x) + 1 & x > 0 \\ g(x) = e^{x-1} & x \in R \end{cases}$

$x = \log_e(g(x)) + 1 \rightarrow g(x) = e^{x-1}$

Question 7

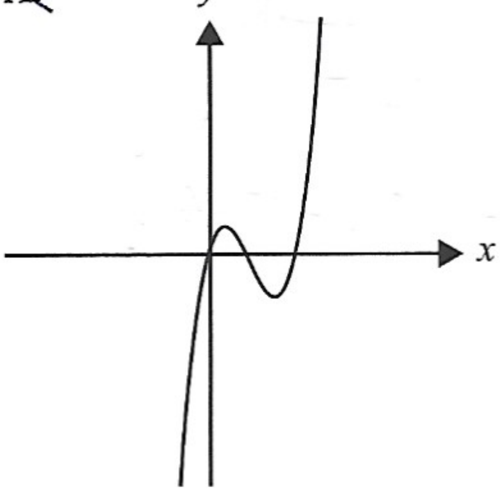
The graph of $y = f(x)$ is shown below.



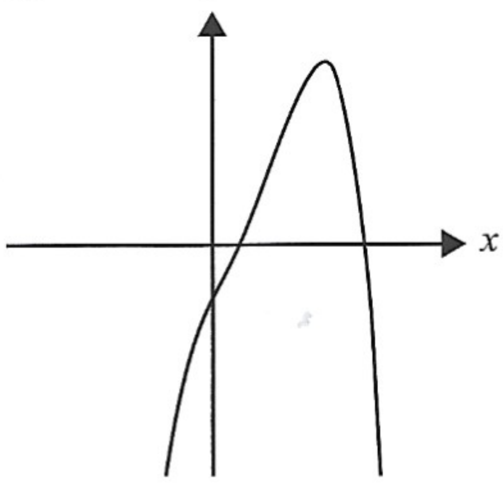
gradient

The graph of $y = f'(x)$, the first derivative of $f(x)$ with respect to x, could be

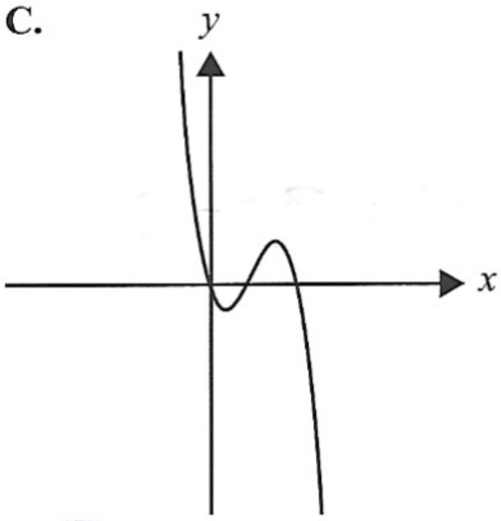
A.



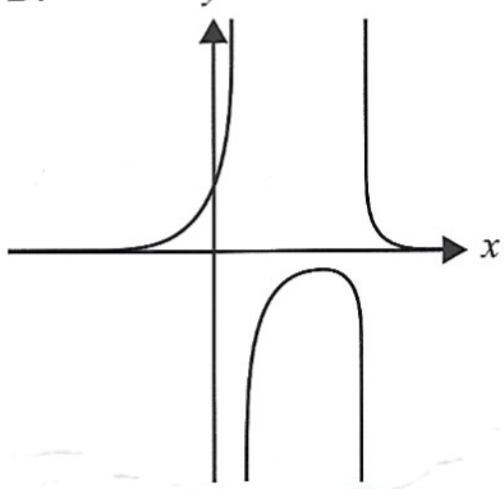
B.



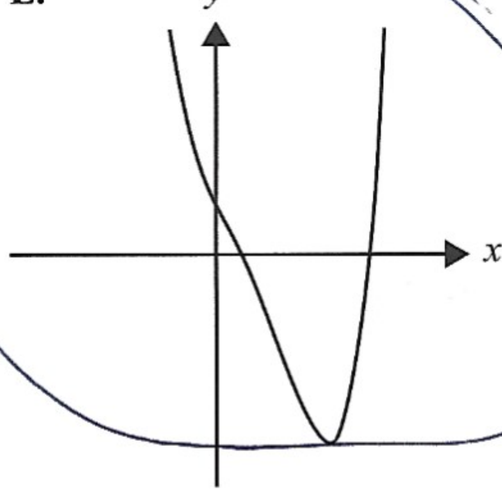
C.



D.



E.



Question 8

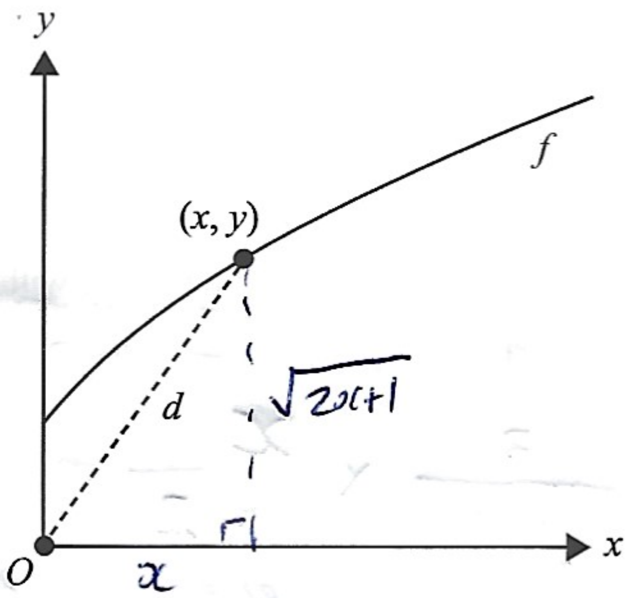
If $\int_0^b f(x)\,dx = 10$ and $\int_0^a f(x)\,dx = -4$, where $0 < a < b$, then $\int_a^b f(x)\,dx$ is equal to

A. -6
B. -4
C. 0
D. 10
E. 14

$\int_a^b f(x)\,dx = \int_0^b f(x)\,dx - \int_0^a f(x)\,dx$

$= 10 - {}^-4 = 14$

Question 9



$d = \sqrt{(x^2) + (\sqrt{2x+1})^2}$

$= \sqrt{x^2 + 2x + 1}$

$= \sqrt{(x+1)^2}$

$= x + 1.$

Let $f : [0, \infty) \to R$, $f(x) = \sqrt{2x+1}$.

The shortest distance, d, from the origin to the point (x, y) on the graph of f is given by

A. $d = x^2 + 2x + 1$
B. $d = x^2 + \sqrt{2x+1}$
C. $d = \sqrt{x^2 - 2x + 1}$
D. $d = x + 1$
E. $d = 2x + 1$

Question 10

An organisation randomly surveyed 1000 Australian adults and found that 55% of those surveyed were happy with their level of physical activity.

An approximate 95% confidence interval for the percentage of Australian adults who were happy with their level of physical activity is closest to

$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

A. (4.1, 6.9)
B. (50.9, 59.1)
C. (52.4, 57.6)
D. (51.9, 58.1)
E. (45.2, 64.8)

$0.55 - 1.96 \times \sqrt{\frac{0.55 \times 0.45}{1000}}$

$= 0.5197$

$= 51.92\%.$

SECTION A – continued
TURN OVER

Question 11

If $\frac{d}{dx}(x \cdot \sin(x)) = \sin(x) + x \cdot \cos(x)$, then $\frac{1}{k}\int x\cos(x)\,dx$ is equal to

$= \frac{1}{k}\int \left[\frac{d}{dx}(x \cdot \sin(x)) - \sin(x)\right] dx.$

$= \frac{1}{k}\left[x\sin(x) - \int \sin(x)\,dx\right] + c$

A. $k\left(x \cdot \sin(x) - \int \sin(x)\,dx\right) + c$

B. $\frac{1}{k}x \cdot \sin(x) - \int \sin(x)\,dx + c$

C. $\frac{1}{k}\left(x \cdot \sin(x) - \int \sin(x)\,dx\right) + c$

D. $\frac{1}{k}(x \cdot \sin(x) - \sin(x)) + c$

E. $\frac{1}{k}\left(\int x \cdot \sin(x)\,dx - \int \sin(x)\,dx\right) + c$

Question 12

A bag contains three red pens and x black pens. Two pens are randomly drawn from the bag without replacement. The probability of drawing a pen of each colour is equal to

Total Pens $= x+3$

$\underset{\text{Red}}{\frac{3}{x+3}} \times \underset{\text{Black}}{\frac{x}{x+2}}$ OR $\underset{\text{Black}}{\frac{x}{x+3}} \times \underset{\text{Red}}{\frac{3}{x+2}}$

$\frac{3x}{(x+3)(x+2)} + \frac{3x}{(x+3)(x+2)}$

$\frac{6x}{(x+3)(x+2)}$

A. $\frac{6x}{(2+x)(3+x)}$

B. $\frac{3x}{(2+x)(3+x)}$

C. $\frac{x}{2+x}$

D. $\frac{3+x}{(2+x)(3+x)}$

E. $\frac{3+x}{5+2x}$

Question 13

The function $f(x) = \log_e\left(\frac{x+a}{x-a}\right)$, where a is a positive real constant, has the maximal domain

A. $[-a, a]$

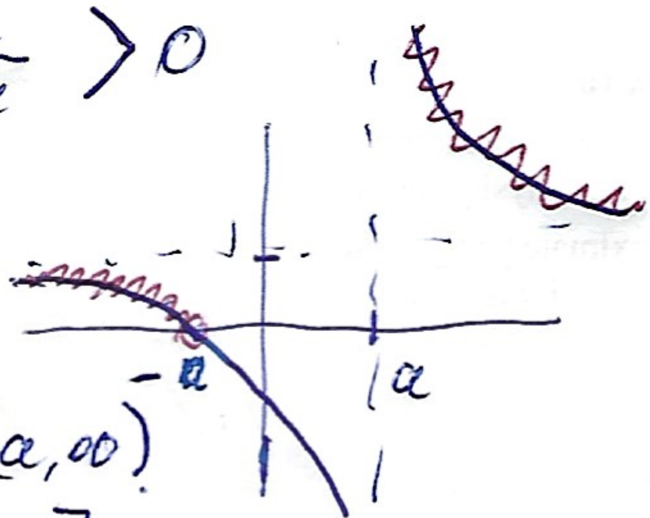
B. $(-a, a)$

C. $R \setminus [-a, a]$

D. $R \setminus (-a, a)$

E. R

$\frac{x+a}{x-a} > 0$



$(-\infty, -a) \cup (a, \infty)$

Same as $R \setminus [-a, a]$

SECTION A – continued

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Question 14

A continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases}\frac{2}{9}xe^{-\frac{1}{9}x^2} & x\geq 0\\ 0 & x<0\end{cases}$$

Use 999 in Calculator

$\int_0^{\infty} x \times \frac{2}{9} x e^{-\frac{1}{9}x^2}\,dx$

$= 2.658680776$

The expected value of X, correct to three decimal places, is

A. 1.000
B. 2.659
C. 3.730
D. 6.341
E. 9.000

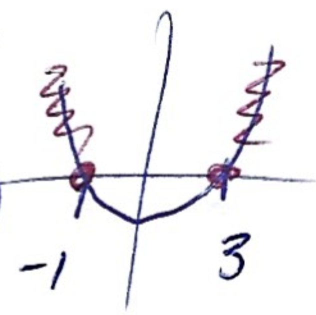
Question 15

The maximal domain of the function with rule $f(x)=\sqrt{x^2-2x-3}$ is given by

A. $(-\infty, \infty)$
B. $(-\infty, -3) \cup (1, \infty)$
C. $(-1, 3)$
D. $[-3, 1]$
E. $(-\infty, -1] \cup [3, \infty)$

$x^2-2x-3 \geq 0$

$(-\infty, -1] \cup [3, \infty)$



Question 16

The function $f(x)=\frac{1}{3}x^3+mx^2+nx+p$, for $m, n, p \in R$, has turning points at $x=-3$ and $x=1$ and passes through the point $(3, 4)$.

The values of m, n and p respectively are

A. $m=0, \quad n=-\frac{7}{3}, \quad p=2$
B. $m=1, \quad n=-3, \quad p=-5$
C. $m=-1, \quad n=-3, \quad p=13$
D. $m=\frac{5}{4}, \quad n=\frac{3}{2}, \quad p=-\frac{83}{4}$
E. $m=\frac{5}{2}, \quad n=6, \quad p=-\frac{91}{2}$

$f'(x) = x^2+2mx+n$.

$f'(-3)$ $\quad 0 = (-3)^2+2\times m\times -3+n$
$0 = 9-6m+n$.

$f'(1)$ $\quad 0 = 1^2+2\times m\times 1+n$
$0 = 1+2m+n$.

$f(3)$ $\quad 4 = \frac{1}{3}(3)^3+m\times 3^2+n\times 3+p$
$4 = 9+9m+3n+p$.

$m=1, n=-3, p=-5$

Question 17

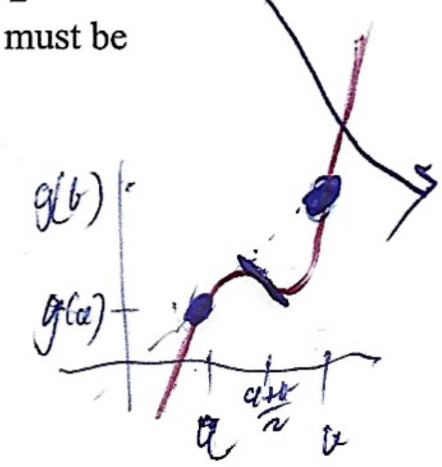
A function g is continuous on the domain $x \in [a, b]$ and has the following properties:

- The average rate of change of g between $x = a$ and $x = b$ is positive. $\rightarrow \dfrac{g(b)-g(a)}{b-a} > 0$
- The instantaneous rate of change of g at $x = \dfrac{a+b}{2}$ is negative.

$\Rightarrow g(b) > g(a)$.

Therefore, on the interval $x \in [a, b]$, the function must be

A. many-to-one.
B. one-to-many.
C. one-to-one.
D. strictly decreasing.
E. strictly increasing.



$\rightarrow g'(x) < 0$ when $x = \dfrac{a+b}{2}$.

Question 18

If X is a binomial random variable where $n = 20$, $p = 0.88$ and $\Pr(X \geq 16 \mid X \geq a) = 0.9175$, correct to four decimal places, then a is equal to

A. 11
B. 12
C. 13
D. 14
E. 15

$\Pr(X \geq 16 \mid X \geq a) = \dfrac{\Pr(X \geq 16 \cap X \geq a)}{\Pr(X \geq a)} = 0.9175$

$\Pr(X \geq 16) = 0.917280$

$= \dfrac{\Pr(X \geq 16)}{\Pr(X \geq a)} = 0.9175$

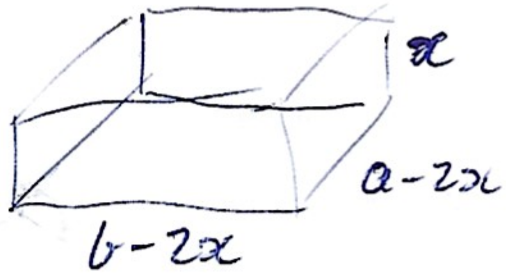
$\dfrac{\Pr(X \geq 16)}{\Pr(X \geq 11)} = 0.9173$ ×

$\dfrac{\Pr(X \geq 16)}{\Pr(X \geq 12)} = 0.9175$ ✓

Question 19

A box is formed from a rectangular sheet of cardboard, which has a width of a units and a length of b units, by first cutting out squares of side length x units from each corner and then folding upwards to form a container with an open top.

The maximum volume of the box occurs when x is equal to



A. $\dfrac{a - b + \sqrt{a^2 - ab + b^2}}{6}$

B. $\dfrac{a + b + \sqrt{a^2 - ab + b^2}}{6}$

C. $\dfrac{a - b - \sqrt{a^2 - ab + b^2}}{6}$

D. $\dfrac{a + b - \sqrt{a^2 - ab + b^2}}{6}$

E. $\dfrac{a + b - \sqrt{a^2 - 2ab + b^2}}{6}$

$Vol = (a-2x)(b-2x)x$

$Vol = 4x^3 - 2ax^2 - 2bx^2 + abx$.

$\dfrac{dV}{dx} = 12x^2 - 4ax - 4bx + ab$.

Solve $\dfrac{dV}{dx} = 0$ for x. Use Calculator

2 Solutions given one not in the domain of x

$x = \dfrac{a+b-\sqrt{a^2+b^2-ab}}{6}$, $x = \cancel{\dfrac{a+b+\sqrt{a^2+b^2-ab}}{6}}$

DO NOT WRITE IN THIS AREA

Question 20

A soccer player kicks a ball with an angle of elevation of $\theta°$, where θ is a normally distributed random variable with a mean of 42° and a standard deviation of 8°.

The horizontal distance that the ball travels before landing is given by the function $d = 50\sin(2\theta)$.

The probability that the ball travels more than 40 m horizontally before landing is closest to

A. 0.969

B. 0.937

C. 0.226

D. 0.149

E. 0.027

$\mu = 42°$ $\sigma = 8°$

$d = 50\sin(2\theta)$.

$\Pr(d > 40)$

for $d = 40$

Solve $40 = 50\sin(2\theta)$

$\theta = 26.565°, 63.435°$

Want $\Pr(26.565 \leq \theta \leq 63.435)$

$= 0.9694698506.$

END OF SECTION A
TURN OVER

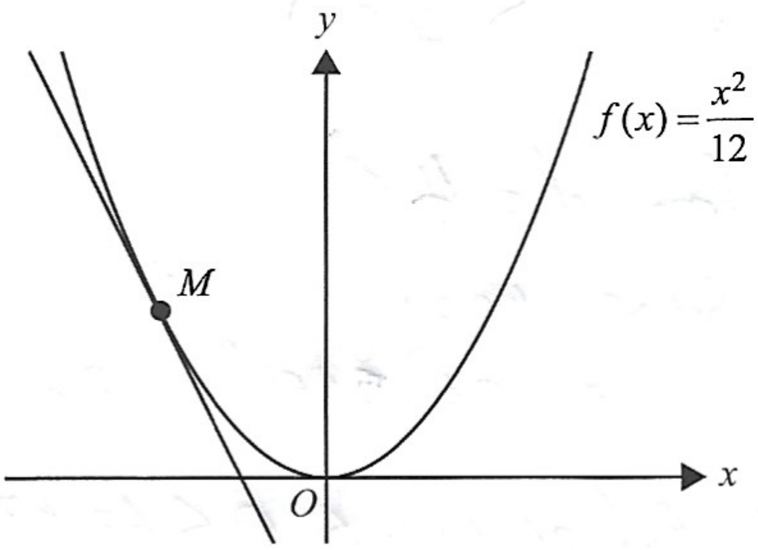
SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

The diagram below shows part of the graph of $y = f(x)$, where $f(x) = \frac{x^2}{12}$.



a. State the equation of the axis of symmetry of the graph of f. 1 mark

$x = 0$

b. State the derivative of f with respect to x. 1 mark

$f'(x) = \frac{x}{6}$

The tangent to f at point M has gradient -2.

c. Find the equation of the tangent to f at point M. 2 marks

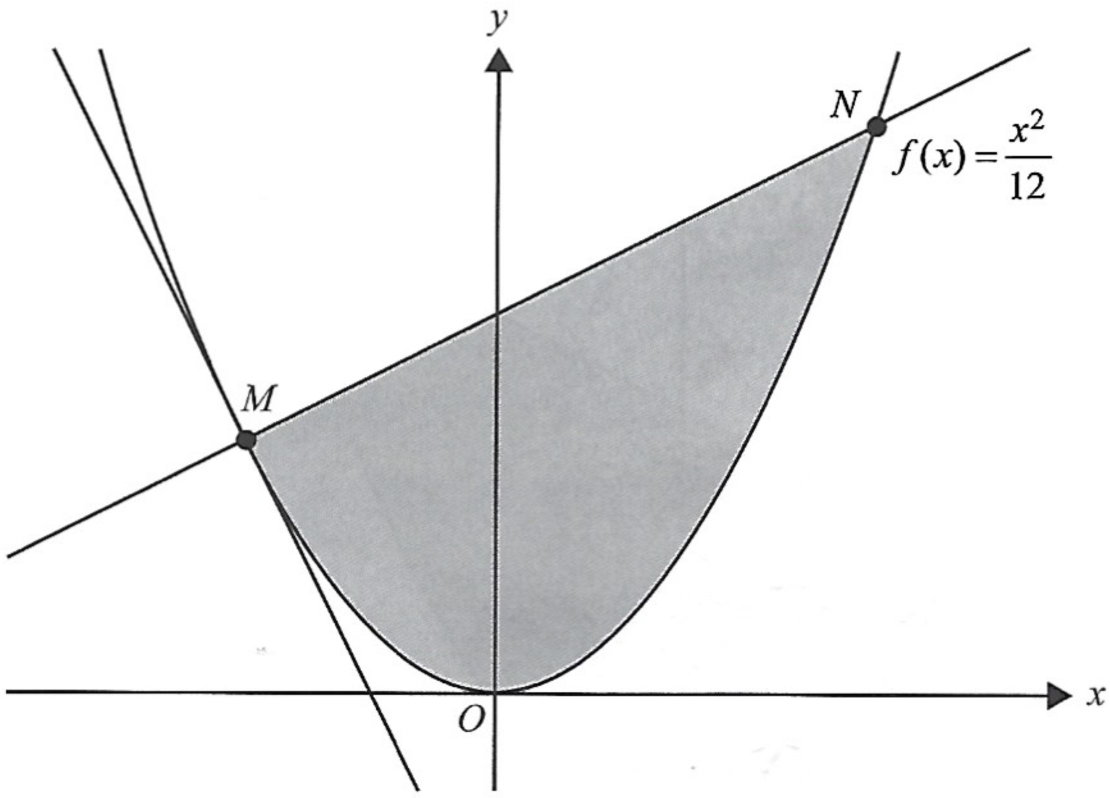
$f'(x) = -2$ $\quad f(-12) = \frac{(-12)^2}{12}$ $\quad y - 12 = -2(x+12)$

$\frac{x}{6} = -2$ $\quad = 12$ $\quad y = -2x - 24 + 12$

$x = -12.$ $\quad y = -2x - 12$

OR Calc -> Interactive -> Calculation -> line -> tanLine

The diagram below shows part of the graph of $y = f(x)$, the tangent to f at point M and the line perpendicular to the tangent at point M.



d. **i.** Find the equation of the line perpendicular to the tangent passing through point M. 1 mark

$y = \frac{x}{2} + 18$

Calc → Interactive → Calculation → line → Normal

ii. The line perpendicular to the tangent at point M also cuts f at point N, as shown in the diagram above.

Find the area enclosed by this line and the curve $y = f(x)$. 2 marks

Point N – (18, 27) Point M – (–12, 12).

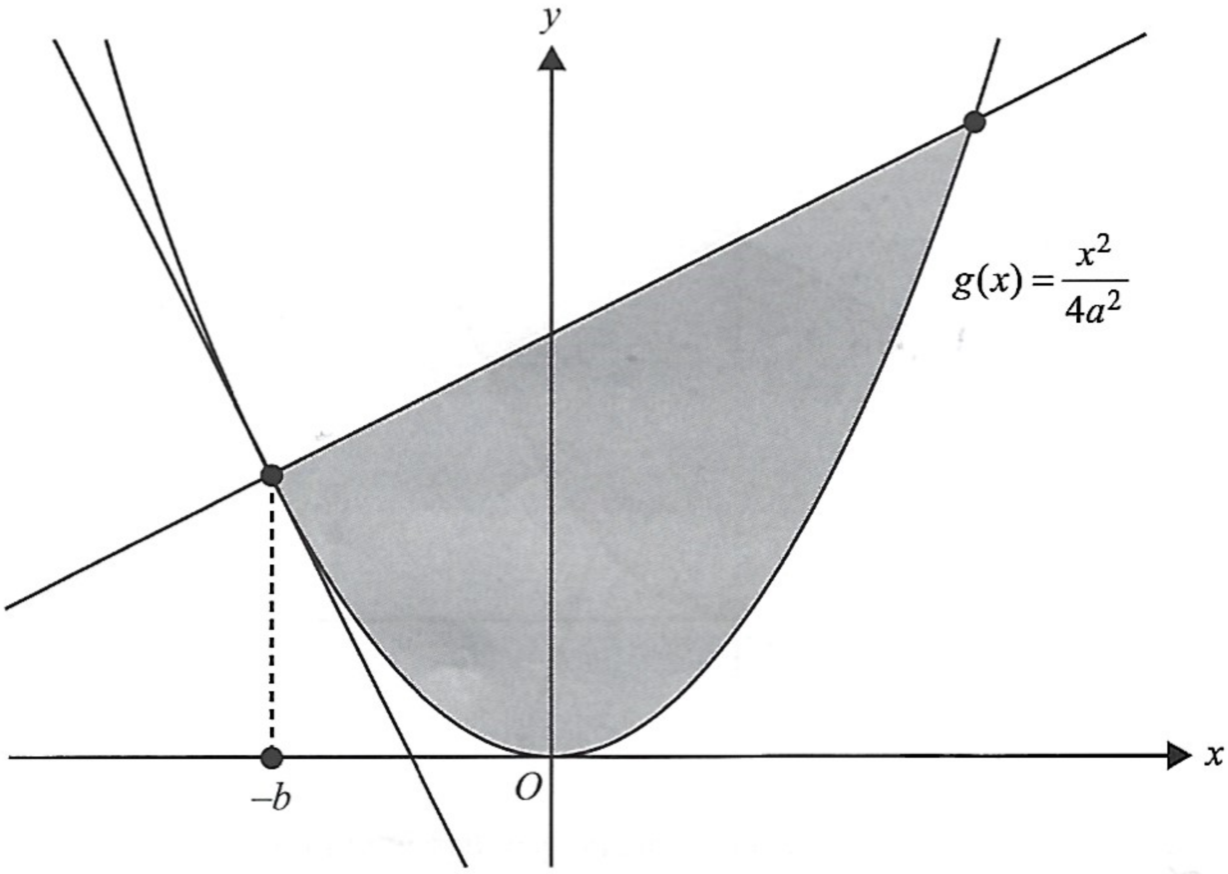
$$\text{Area} = \int_{-12}^{18} \left(\frac{x}{2} + 18\right) dx - \int_{-12}^{18} \left(\frac{x^2}{12}\right) dx.$$

$$= 375$$

Use Calculator – Solve $f(x) = \frac{x^2}{12}$ and $y = \frac{x}{2} + 18$ simultaneously to find the points of intersection thus the limits on the integral.

e. Another parabola is defined by the rule $g(x) = \frac{x^2}{4a^2}$, where $a > 0$.

A tangent to g and the line perpendicular to the tangent at $x = -b$, where $b > 0$, are shown below.



Find the value of b, in terms of a, such that the shaded area is a minimum. 4 marks

Normal at $x = -b$. Use Calulator

$$y_N = \frac{2a^2}{b}x + 2a^2 + \frac{b^2}{4a^2}$$

$g(x) = y_N$ at $\left(-b, \frac{b^2}{4a^2}\right)$ $\left(\frac{8a^4}{b} + b, \frac{(8a^4+b^2)^2}{4a^2b^2}\right)$.

$$\text{Area} = \int_{-b}^{\frac{8a^4}{b}+b} \left(\frac{2a^2}{b}x + 2a^2 + \frac{b^2}{4a^2} - \frac{x^2}{4a^2}\right)dx$$

Use Calculator and Simplify answer.

$$= \frac{(4a^4+b^2)^3}{3a^2b^3}$$

Area a min when $b = 2a^2$. other answers given 2 are not real 1 b is on the +ve x-axis

To find b use Calc → Solve $\left(\frac{d}{db}\left(\frac{(4a^4+b^2)^3}{3a^2b^3}\right) = 0, b\right)$ See next Page for More.

Note: Need to be confident with the calculator functions and being able to Copy and Paste

DO NOT WRITE IN THIS AREA

Calculator Functions which are used in Question 1.

$\frac{d}{d\square}\square$, $\int_{\square}^{\square}\square\, d\square$

tan Line , normal

Simultaneous Equations $\begin{cases}\square\\\square\end{cases}\Big|\square$

Solve , Simplify

Copy and Paste equations.

CONTINUES OVER PAGE

Solve $\left(\frac{d}{db}\left(\frac{(4a^4+b^2)^3}{3a^2b^3}\right)=0, b\right)$.

$b=-2\sqrt{-a^4}$, $b=2\sqrt{-a^4}$, $b=-2a^2$, $b=2a^2$

Not REAL

Places the left intersection on the +ve x-axis

This is the Solution we are after

The question asks for b. Not $-b$.

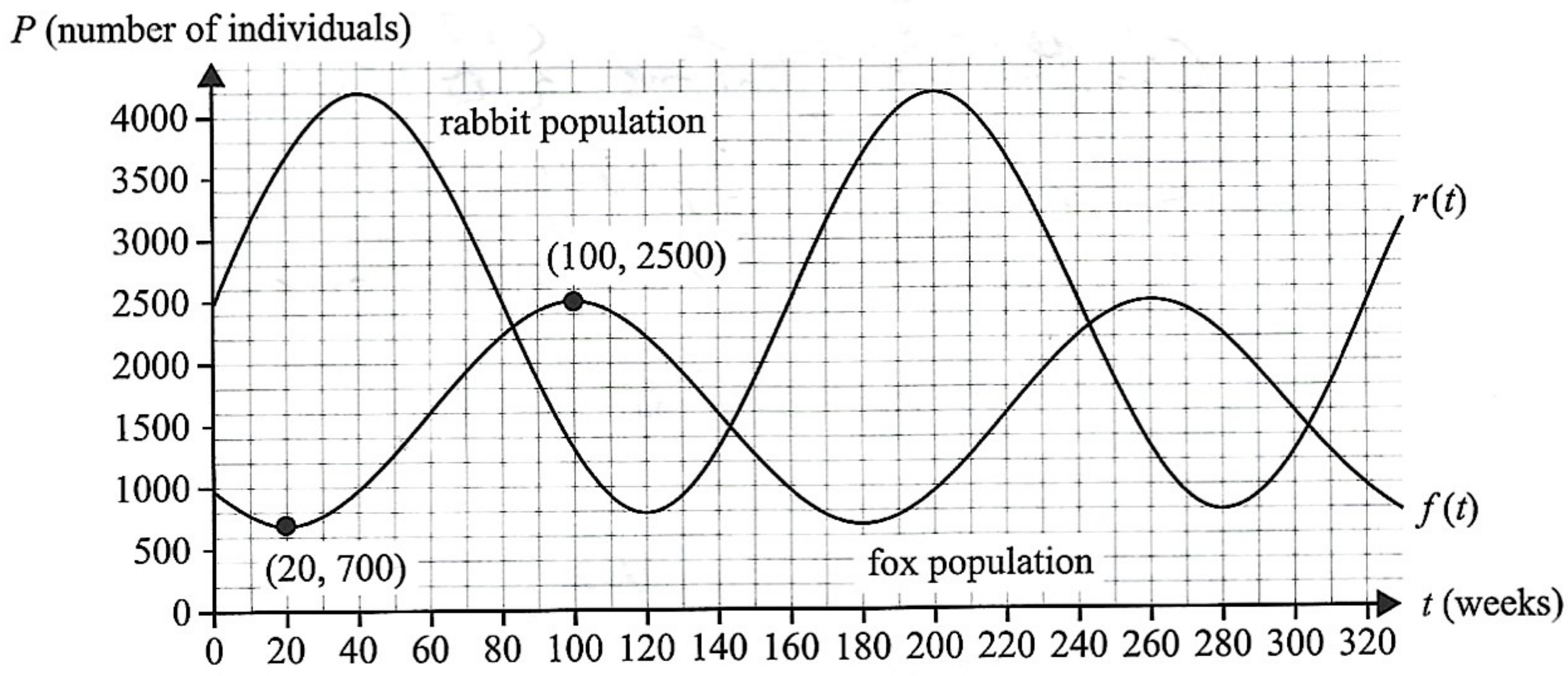
Question 2 (16 marks)

On a remote island, there are only two species of animals: foxes and rabbits. The foxes are the predators and the rabbits are their prey.

The populations of foxes and rabbits increase and decrease in a periodic pattern, with the period of both populations being the same, as shown in the graph below, for all $t \geq 0$, where time t is measured in weeks.

One point of minimum fox population, (20, 700), and one point of maximum fox population, (100, 2500), are also shown on the graph.

The graph has been drawn to scale.



The population of rabbits can be modelled by the rule $r(t) = 1700\sin\left(\frac{\pi t}{80}\right) + 2500$.

a. **i.** State the initial population of rabbits. 1 mark

2500

ii. State the minimum and maximum population of rabbits. 1 mark

800 4200

iii. State the number of weeks between maximum populations of rabbits. 1 mark

200 − 40 = 160

Need some working out. Simultaneous Eq^{ns} or Equate amplitude and Period

The population of foxes can be modelled by the rule $f(t) = a\sin(b(t-60)) + 1600$.

b. Show that $a = 900$ and $b = \frac{\pi}{80}$. 2 marks

amplitude - $a = \frac{2500-700}{2} = 900$.

Period - $\frac{2\pi}{b} = 160$

$b = \frac{2\pi}{160} = \frac{\pi}{80}$

OR Simultaneous Eq^{ns}.

$700 = a\sin(b(20-60)) + 1600$

$2500 = a\sin(b(100-60)) + 1600$

Solve using Calculator.

c. Find the maximum combined population of foxes and rabbits. Give your answer correct to the nearest whole number. 1 mark

Could graph then find the maximum

Max of $r(t) + f(t)$ $\quad 5339.5 \Rightarrow 5339$

Can't have half animal → round DOWN

d. What is the number of weeks between the periods when the combined population of foxes and rabbits is a maximum? 1 mark

$213.73 - 53.73 = 160$

The population of foxes is better modelled by the transformation of $y = \sin(t)$ under Q given by

$$Q: R^2 \to R^2, Q\left(\begin{bmatrix} t \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{90}{\pi} & 0 \\ 0 & 900 \end{bmatrix}\begin{bmatrix} t \\ y \end{bmatrix} + \begin{bmatrix} 60 \\ 1600 \end{bmatrix}$$

e. Find the average population during the first 300 weeks for the combined population of foxes and rabbits, where the population of foxes is modelled by the transformation of $y = \sin(t)$ under the transformation Q. Give your answer correct to the nearest whole number. 4 marks

$t' = \frac{90}{\pi} t + 60$ $\quad\quad y' = 900y + 1600$

$t = \frac{\pi(t'-60)}{90}$ $\quad\quad y = \frac{y'-1600}{900}$

$\frac{y'-1600}{900} = \sin\left(\frac{\pi(t'-60)}{90}\right)$

$y' = 900\sin\left(\frac{\pi(t'-60)}{90}\right) + 1600$

Total Population in first 300 weeks $= \int_0^{300} (y' + r(t))\,dt$

Average Population $= \frac{1}{300}\int_0^{300} (y' + r(t))\,dt$. ← Calculator

$= 4142.26$.

$= 4142$.

SECTION B – Question 2 – continued
TURN OVER

Careful when entering equations into Calculator.

Over a longer period of time, it is found that the increase and decrease in the population of rabbits gets smaller and smaller.

The population of rabbits over a longer period of time can be modelled by the rule

$$s(t) = 1700 \cdot e^{-0.003t} \cdot \sin\left(\frac{\pi t}{80}\right) + 2500, \qquad \text{for all } t \geq 0$$

f. Find the average rate of change between the first two times when the population of rabbits is at a maximum. Give your answer correct to one decimal place. 2 marks

$(38.058, 4012.2)$

$(198.058, 3435.7)$

$$\text{rate} = \frac{4012.2 - 3435.7}{38.058 - 198.058}$$
$$= -3.6031$$
$$\approx -3.6$$

g. Find the time, where $t > 40$, in weeks, when the rate of change of the rabbit population is at its greatest positive value. Give your answer correct to the nearest whole number. 2 marks

$$t = 156.11$$
$$= 156$$

h. Over time, the rabbit population approaches a particular value.

State this value. 1 mark

2500

Note: Part g. Can graph $\frac{ds}{dt}$ and find the first local Maximum

OR can graph $\frac{d^2s}{dt^2}$ look for $= 0$

OR Solve $\frac{d^2s}{dt^2} = 0$.

Part h. directly from the equation ..... $+2500$

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Question 3 (14 marks)

Mika is flipping a coin. The unbiased coin has a probability of $\frac{1}{2}$ of landing on heads and $\frac{1}{2}$ of landing on tails.

Let X be the binomial random variable representing the number of times that the coin lands on heads.

Mika flips the coin five times.

a. **i.** Find $\Pr(X = 5)$. 1 mark

$= \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

Note: EXACT answer needed for a i and a ii and a iv

ii. Find $\Pr(X \geq 2)$. 1 mark

$= 1-(Pr(X=0)+Pr(X=1))$

$= 1-\left(1\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 + 5\times\left(\frac{1}{2}\right)^4\times\left(\frac{1}{2}\right)^1\right) = 1-\left(\frac{1}{32}+\frac{5}{32}\right) = \frac{13}{16}$

iii. Find $\Pr(X \geq 2 \mid X < 5)$, correct to three decimal places. 2 marks

$= \frac{Pr(X \geq 2 \cap X < 5)}{Pr(X<5)} = \frac{Pr(2 \leq X \leq 4)}{Pr(X \leq 4)}$

$= \frac{0.78125}{0.96875}$

$= 0.806$

iv. Find the expected value and the standard deviation for X. 2 marks

$E(X) = np$

$= 5 \times \frac{1}{2}$

$= \frac{5}{2}$

$sd(X) = \sqrt{np(1-p)}$

$= \sqrt{5 \times \frac{1}{2} \times (1-\frac{1}{2})}$

$= \frac{\sqrt{5}}{2}$.

The height reached by each of Mika's coin flips is given by a continuous random variable, H, with the probability density function

$$f(h)=\begin{cases} ah^2+bh+c & 1.5 \le h \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

where h is the vertical height reached by the coin flip, in metres, between the coin and the floor, and a, b and c are real constants.

b. **i.** State the value of the definite integral $\int_{1.5}^{3} f(h)\,dh$. 1 mark

Integral is whole domain thus Probability = 1.

$= 1$

ii. Given that $\Pr(H \le 2) = 0.35$ and $\Pr(H \ge 2.5) = 0.25$, find the values of a, b and c. 3 marks

$\int_{1.5}^{3} f(h)\,dh = 1$, $\int_{1.5}^{2} f(h)\,dh = 0.35$, $\int_{2.5}^{3} f(h)\,dh = 0.25$

$a = -\frac{4}{5}$ $b = \frac{17}{5}$ $c = -\frac{167}{60}$

Solve the integrals simultaneously on Calculator
Exact answer required.

iii. The ceiling of Mika's room is 3 m above the floor. The minimum distance between the coin and the ceiling is a continuous random variable, D, with probability density function g.
The function g is a transformation of the function f given by $g(d) = f(rd + s)$, where d is the minimum distance between the coin and the ceiling, and r and s are real constants.

Find the values of r and s. 1 mark

$r = -1$ $s = 3$

Think about the transformation from f to g.
f has the height - measured from floor
g has the distance - measured from the ceiling
Thus taking away from the 3 m ceiling

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c. Mika's sister Bella also has a coin. On each flip, Bella's coin has a probability of p of landing on heads and $(1-p)$ of landing on tails, where p is a constant value between 0 and 1.

Bella flips her coin 25 times in order to estimate p.

Let $\hat{P}$ be the random variable representing the proportion of times that Bella's coin lands on heads in her sample.

i. Is the random variable $\hat{P}$ discrete or continuous? Justify your answer. 1 mark

Discrete. Whole number of times, countable.

ii. If $\hat{p} = 0.4$, find an approximate 95% confidence interval for p, correct to three decimal places. 1 mark

$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ Formula Sheet.

$\left(0.4 - 1.96 \times \sqrt{\frac{0.4 \times 0.6}{25}}, 0.4 + 1.96\sqrt{\frac{0.4 \times 0.6}{25}}\right)$

$(0.207, 0.592)$

iii. Bella knows that she can decrease the width of a 95% confidence interval by using a larger sample of coin flips.

If $\hat{p} = 0.4$, how many coin flips would be required to halve the width of the confidence interval found in **part c.ii.**? 1 mark

New $Sd(\hat{p}) = \sqrt{\frac{0.4 \times 0.6}{25}} \times \frac{1}{2} = \sqrt{\frac{0.4 \times 0.6}{25 \times 4}} = \sqrt{\frac{0.4 \times 0.6}{100}}$

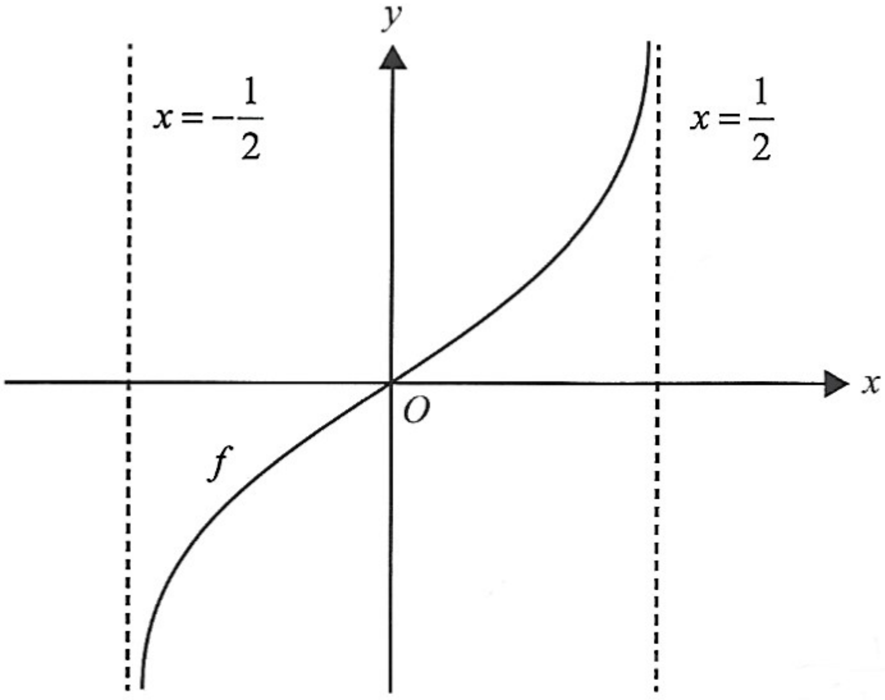
100 coin flips.

Note: width of 95% confidence interval determined by $Sd(\hat{p})$ $\Rightarrow$ to halve the width need to halve the $Sd(\hat{p})$

Question 4 (10 marks)

Consider the function f, where $f:\left(-\frac{1}{2},\frac{1}{2}\right)\to R$, $f(x)=\log_e\left(x+\frac{1}{2}\right)-\log_e\left(\frac{1}{2}-x\right)$.

Part of the graph of $y=f(x)$ is shown below.



y-values.

a. State the range of $f(x)$. 1 mark

R.

b. **i.** Find $f'(0)$. 2 marks

$f'(x) = -\frac{4}{(2x+1)(2x-1)}$

$f'(0) = 4$

ii. State the maximal domain over which f is strictly increasing. 1 mark

$\left(-\frac{1}{2}, \frac{1}{2}\right)$

Working out required

c. Show that $f(x) + f(-x) = 0$. 1 mark

$f(x)+f(-x) = \cancel{\log_e\left(x+\frac{1}{2}\right)} - \cancel{\log_e\left(\frac{1}{2}-x\right)} + \cancel{\log_e\left(-x+\frac{1}{2}\right)} - \cancel{\log_e\left(\frac{1}{2}+x\right)}$

$= 0.$

d. Find the domain and the rule of f^{-1}, the inverse of f. 3 marks

Let $y = f(x)$

$y = \log_e\left(x+\frac{1}{2}\right) - \log_e\left(\frac{1}{2}-x\right)$

Inverse -

$x = \log_e\left(y+\frac{1}{2}\right) - \log_e\left(\frac{1}{2}-y\right)$ ← Use calculator Solve for y.

$y = \dfrac{e^x-1}{2(e^x+1)}$

$f^{-1}(x) = \dfrac{e^x-1}{2(e^x+1)}.$ Domain R.

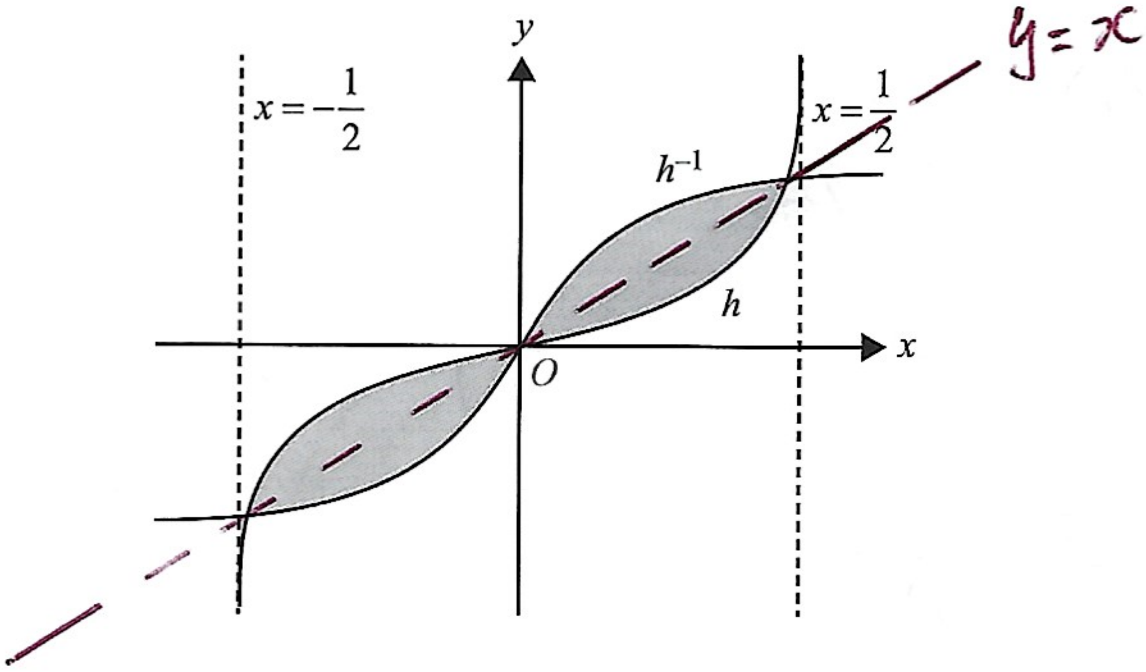
Must have final answer as $f^{-1}(x)$

e. Let h be the function $h: \left(-\frac{1}{2}, \frac{1}{2}\right) \to R$, $h(x) = \frac{1}{k}\left(\log_e\left(x+\frac{1}{2}\right) - \log_e\left(\frac{1}{2}-x\right)\right)$, where $k \in R$ and $k > 0$.

The inverse function of h is defined by $h^{-1}: R \to R$, $h^{-1}(x) = \dfrac{e^{kx}-1}{2(e^{kx}+1)}$.

The area of the regions bound by the functions h and h^{-1} can be expressed as a function, $A(k)$.

The graph below shows the relevant area shaded.



You are not required to find or define $A(k)$.

i. Determine the range of values of k such that $A(k) > 0$. 1 mark

$k > 4$

ii. Explain why the domain of $A(k)$ does not include all values of k. 1 mark

If $h(x) \geq h^{-1}(x)$ for $x > 0$ there will be no bounded area. The same applies if $h(x) \leq h^{-1}(x)$ for $x < 0$.

Note: $h(x)$ and $h^{-1}(x)$ intersect on the line $y = x$.
For a bounded area to exist there must be two points of intersection between $h(x)$ and $h^{-1}(x)$ in the domain $0 \leq x < \frac{1}{2}$. One will be $x = 0$. Need another.
The same applies for the intersection between $h(x)$ and $y = x$

PTO.

On Calculator

Solve $\left(\frac{1}{k}\left(\log_e\left(x+\frac{1}{2}\right)-\log_e\left(\frac{1}{2}-x\right)\right)=x,\ x\right)$.

try $k=1, 2, 3, 4, 5$.

$k=1,2,3,4$ give one Solution $x=0$.

$k=5$ gives 2 solutions, $0\leq x<\frac{1}{2}$.

Thus ~~for~~ a bounded area $k>4$.

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Question 5 (9 marks)

Consider the composite function $g(x) = f(\sin(2x))$, where the function $f(x)$ is an unknown but differentiable function for all values of x.

Use the following table of values for f and f'.

x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$f(x)$	-2	5	3
$f'(x)$	7	0	$\frac{1}{9}$

a. Find the value of $g\left(\frac{\pi}{6}\right)$. 1 mark

$g\left(\frac{\pi}{6}\right) = f\left(\sin\left(2 \times \frac{\pi}{6}\right)\right)$

$= f\left(\sin\left(\frac{\pi}{3}\right)\right)$

$= f\left(\frac{\sqrt{3}}{2}\right) = 3.$

The derivative of g with respect to x is given by $g'(x) = 2 \cdot \cos(2x) \cdot f'(\sin(2x))$.

b. Show that $g'\left(\frac{\pi}{6}\right) = \frac{1}{9}$. 1 mark

$g'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) \cdot f'\left(\frac{\sqrt{3}}{2}\right)$

$= 2 \times \frac{1}{2} \times \frac{1}{9} = \frac{1}{9}$

c. Find the equation of the tangent to g at $x = \frac{\pi}{6}$. 2 marks

at $x = \frac{\pi}{6}$ $g\left(\frac{\pi}{6}\right) = 3$ $g'\left(\frac{\pi}{6}\right) = \frac{1}{9}$

$y - 3 = \frac{1}{9}\left(x - \frac{\pi}{6}\right)$

$y - 3 = \frac{x}{9} - \frac{\pi}{54}$

$y = \frac{x}{9} - \frac{\pi}{54} + 3$.

↙ Average Value formula from Bound reference.

d. Find the average value of the derivative function $g'(x)$ between $x = \frac{\pi}{8}$ and $x = \frac{\pi}{6}$. 2 marks

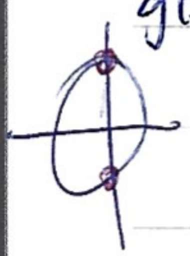
$$\frac{1}{\frac{\pi}{6}-\frac{\pi}{8}} \int_{\frac{\pi}{6}}^{\frac{\pi}{8}} \left(g'(x)\right) dx = \frac{24}{\pi}\left[g(x)\right]_{\frac{\pi}{8}}^{\frac{\pi}{6}}$$

$$= \frac{24}{\pi}\left(g\left(\frac{\pi}{6}\right) - g\left(\frac{\pi}{8}\right)\right)$$

$$= \frac{24}{\pi}(3-5) = \frac{24}{\pi} \times -2 = -\frac{48}{\pi}$$

e. Find **four** solutions to the equation $g'(x) = 0$ for the interval $x \in [0, \pi]$. 3 marks

$g'(x) = 2\cos(2x)\, f'(\sin(2x)) = 0.$



$\cos(2x) = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}$

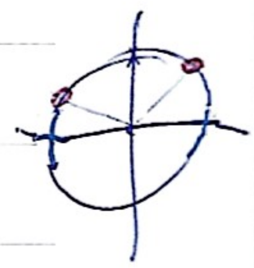
$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$f'(\sin(2x)) = 0$

$\sin(2x) = \frac{\sqrt{2}}{2}.$

$2x = \frac{\pi}{4}, \frac{3\pi}{4}$

$x = \frac{\pi}{8}, \frac{3\pi}{8}.$



$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{8}, \frac{3\pi}{8}.$

Note: for d. $g\left(\frac{\pi}{8}\right) = f\left(\sin\left(2 \times \frac{\pi}{8}\right)\right)$
$= f\left(\sin\left(\frac{\pi}{4}\right)\right)$
$= f\left(\frac{\sqrt{2}}{2}\right)$
$= 5$

for e. from table $f'(x) = 0$ when $x = \frac{\sqrt{2}}{2}$
$\Rightarrow \sin(2x) = \frac{\sqrt{2}}{2}.$

END OF QUESTION AND ANSWER BOOK