VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2022

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

ANS - SYNNO

STUDENT NUMBER									Letter

SPECIALIST MATHEMATICS
Written examination 1

Friday 4 November 2022

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 13 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (3 marks)

Consider the equation $p(z) = z^2 + 6iz - 25$, $z \in C$.

Note: Required form for the solution

a. Express $p(z)$ in the form $p(z) = (z + ai)^2 + b$, where $a, b \in R$. 1 mark

$P(z) = z^2 + 6iz - 25$

$= (z^2 + 6iz + 9i^2) - 9i^2 - 25$

$= (z + 3i)^2 + 9 - 25$

$P(z) = (z + 3i)^2 - 16.$

b. Hence, or otherwise, find the solutions of the equation $p(z) = 0$. 2 marks

$0 = (z + 3i)^2 - 16$

$0 = (z + 3i - 4)(z + 3i + 4)$

$z = -3i + 4, \quad z = -3i - 4$

Alternative

$0 = (z + 3i)^2 - 16$

$16 = (z + 3i)^2$

$\pm 4 = z + 3i$

$z = -3i \pm 4.$

Can also use quadratic formula.

Question 2 (3 marks)

Solve the differential equation $\frac{dy}{dx} = -x\sqrt{4-y^2}$ given that $y(2) = 0$. Give your answer in the form $y = f(x)$.

$$\int \frac{1}{\sqrt{4-y^2}}\,dy = -\int x\,dx.$$

$$\sin^{-1}\left(\frac{y}{2}\right) = -\frac{1}{2}x^2 + c.$$

$y(2) = 0 \rightarrow$ when $x = 2$ $y = 0$

$$\sin^{-1}\left(\frac{0}{2}\right) = -\frac{1}{2}\times 2^2 + c$$

$$0 = -2 + c$$

$$c = 2$$

$$\sin^{-1}\left(\frac{y}{2}\right) = -\frac{1}{2}x^2 + 2$$

$$\frac{y}{2} = \sin\left(-\frac{1}{2}x^2 + 2\right)$$

$$y = 2\sin\left(-\frac{x^2}{2} + 2\right)$$

$$y = 2\sin\left(\frac{4-x^2}{2}\right)$$

Alternative $\int \frac{-1}{\sqrt{4-y^2}}\,dy = \int x\,dx.$

$$\cos^{-1}\left(\frac{y}{2}\right) = \frac{1}{2}x^2 + c.$$

$x = 2$ $y = 0$

$$\cos^{-1}(0) = \frac{1}{2}\times 2^2 + c.$$

$$\frac{\pi}{2} = 2 + c$$

$$c = \frac{\pi}{2} - 2.$$

$$\cos^{-1}\left(\frac{y}{2}\right) = \frac{1}{2}x^2 + \frac{\pi}{2} - 2.$$

$$\frac{y}{2} = \cos\left(\frac{x^2}{2} + \frac{\pi}{2} - 2\right)$$

$$y = 2\cos\left(\frac{x^2+\pi-4}{2}\right).$$

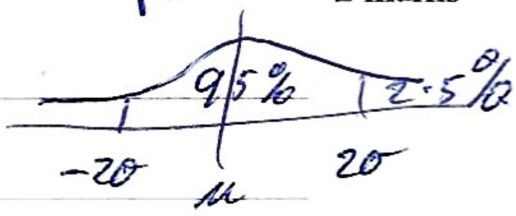
Question 3 (4 marks)

The time taken by a coffee machine to dispense a cup of coffee varies normally with a mean of 10 seconds and a standard deviation of 1.5 seconds.

$E(X) = 10 \quad sd(X) = 1.5$

a. Find the probability that more than 34 seconds is needed to dispense a total of four cups of coffee. Give your answer correct to two decimal places. 2 marks

$4X$



$E(4X) = 4E(X) = 40$

$Var(4X) = 4^2\, Var(X)$

$sd(4X) = 2\, sd(X)$

$= 3.$

$34 = 40 - 6$

$= 40 - 2 \times 3$

$= E(4X) - 2 \times sd(4X)$ ie $\mu - 2\sigma$

$\Pr(4X > 34) = 0.975 = 0.98$

b. The machine is to be serviced. After it is serviced, it is expected that the mean time taken to dispense a cup of coffee will be reduced, but that the standard deviation will remain the same.

Following the service, the mean time taken to dispense 25 cups of coffee is found to be 9 seconds.

$z = 1.96$

Find a 95% confidence interval for the mean time that the machine takes to dispense a cup of coffee following the service. Give your answer in seconds, correct to one decimal place. 2 marks

$n = 25 \qquad \bar{x} = 9.$

$\left(9 - 1.96 \times \frac{1.5}{5},\ 9 + 1.96 \times \frac{1.5}{5}\right).$

$(9 - 0.588,\ 9 + 0.588$

$(9 - 0.6,\ 9 + 0.6)$

$(8.4,\ 9.6).$

Note: - Confidence interval on formula sheet

- Can use $z = 1.96$ or $z = 2$.

- Using $z = 2$ the numbers are 'nicer' to work with

$$5 \overline{)2.940}\quad 0.588$$

$$\begin{array}{r} 1.96 \\ \times\ 1.5 \\ \hline 980 \\ 1960 \\ \hline 2.940 \end{array}$$

Question 4 (4 marks)

Find $\int \frac{3x^2+4x+12}{x(x^2+4)}\,dx$.

$$\frac{3x^2+4x+12}{x(x^2+4)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\equiv \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)}$$

$$\equiv \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}$$

$$\equiv \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}.$$

$4A = 12$ $\quad C = 4 \quad$ $A + B = 3$

$A = 3$ $\quad\quad\quad\quad\quad$ $3 + B = 3$

$B = 0$

$$\int \frac{3x^2+4x+12}{x(x^2+4)}\,dx = \int \left(\frac{3}{x} + \frac{4}{x^2+4}\right) dx$$

$$= 3\int \frac{1}{x}\,dx + 2\int \frac{2}{x^2+4}\,dx$$

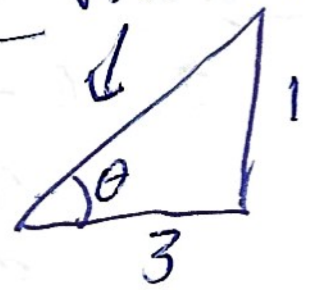
$$= 3\log_e|x| + 2\,\mathrm{Tan}^{-1}\left(\frac{x}{2}\right) + C.$$

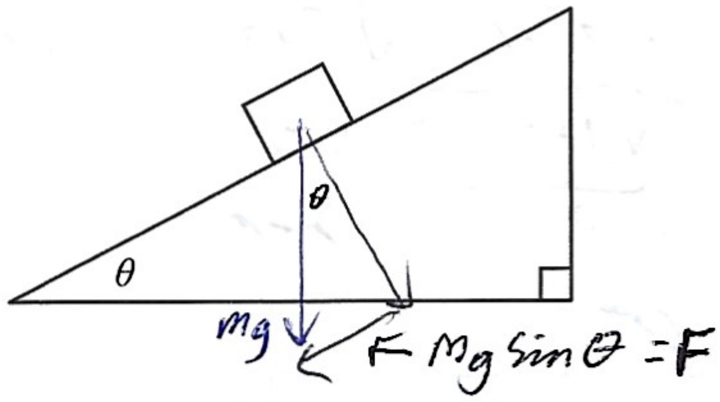
Need these

Question 5 (3 marks)

A body of mass 10 kg, which is initially at rest, slides down a smooth inclined plane, as shown in the diagram below. The plane is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{1}{3}$.

$\sqrt{3^2+1^2} = \sqrt{10}$





a. Find the speed of the body after the body has been in motion for two seconds. 2 marks

$F = 10\,g\sin\theta$

$= 10 \times g \times \frac{1}{\sqrt{10}}$

$= \sqrt{10}\,g$

$a = \frac{F}{m}$

$= \frac{\sqrt{10}\,g}{10}$

$u = 0 \quad a = \frac{\sqrt{10}\,g}{10}, \quad t = 2 \quad V = ?$

$v = u + at$

$v = 0 + \frac{\sqrt{10}\,g}{10} \times 2.$

$v = \frac{\sqrt{10}\,g}{5}.$

b. After the body has been in motion for two seconds, a constant braking force, R newtons, is applied to the body parallel to the plane so that the body has constant velocity.

Find the value of R. 1 mark

Velocity Constant $\Rightarrow a = 0$

$\Rightarrow$ Resultant force $= 0$.

Breaking Force $= R = \sqrt{10}\,g$. up the plane.

Question 6 (6 marks)

a. Find the cosine of the acute angle between the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$. 2 marks

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |a|\,|b| \cos\theta$

$2 - 6 + 12 = \sqrt{2^2 + (-3)^2 + 6^2}\ \sqrt{1^2 + 2^2 + 2^2}\ \cos\theta$

$8 = \sqrt{4+9+36}\ \sqrt{1+4+4}\ \cos\theta$

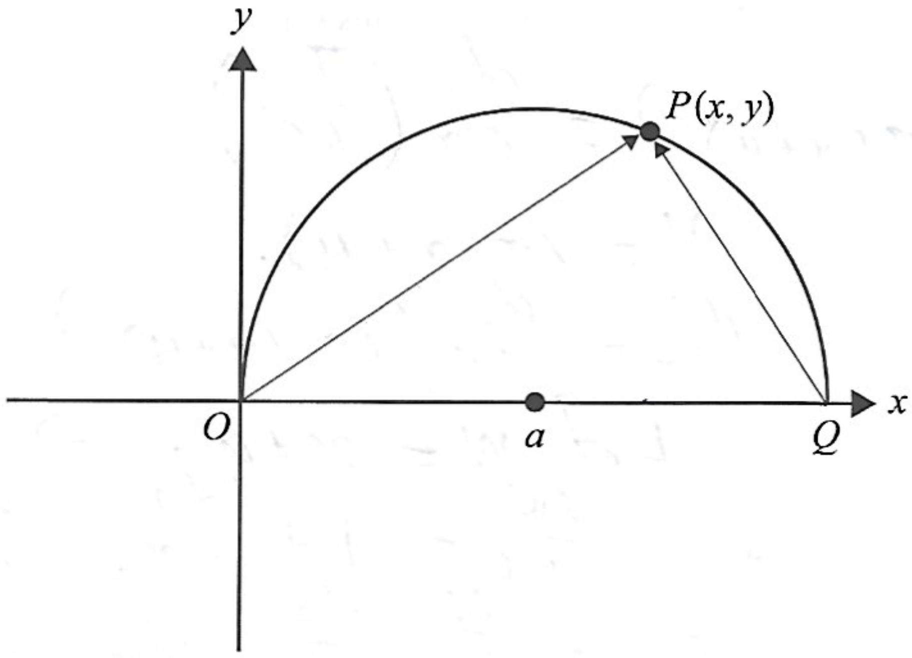
$8 = \sqrt{49}\ \sqrt{9}\ \cos\theta$

$8 = 7 \times 3 \cos\theta$

$8 = 21 \cos\theta$

$\cos\theta = \frac{8}{21}.$

b. OPQ is a semicircle of radius a with equation $y = \sqrt{a^2 - (x-a)^2}$. $P(x, y)$ is a point on the semicircle OPQ, as shown below.



i. Express the vectors $\overrightarrow{OP}$ and $\overrightarrow{QP}$ in terms of $a, x, y, \underset{\sim}{i}$ and $\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a unit vector in the direction of the positive x-axis and $\underset{\sim}{j}$ is a unit vector in the direction of the positive y-axis.

1 mark

$\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$ $\quad \overrightarrow{QP} = (x-2a)\underset{\sim}{i} + \sqrt{a^2-(x-a)^2}\,\underset{\sim}{j}$

$= x\underset{\sim}{i} + \sqrt{a^2-(x-a)^2}\,\underset{\sim}{j}$

ii. Hence, using the vector scalar (dot) product, determine whether $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{QP}$.

3 marks

$\overrightarrow{OP} \cdot \overrightarrow{QP} = x(x-2a) + y^2$

$= x^2 - 2ax + a^2 - (x-a)^2$

$= x^2 - 2ax + a^2 - [x^2 - 2ax + a^2]$

$= [x^2 - 2ax + a^2] - [x^2 - 2ax + a^2]$

$= 0.$

Thus $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{QP}$

Need to make this statement

Note: required form for answer.

Question 7 (3 marks)

A curve has equation $x\cos(x+y)=\dfrac{\pi}{48}$.

Find the gradient of the curve at the point $\left(\dfrac{\pi}{24},\dfrac{7\pi}{24}\right)$. Give your answer in the form $\dfrac{a\sqrt{b}-\pi}{\pi}$, where $a, b \in Z$.

$\frac{dy}{dx}$

Use Product Rule.

$$\rightarrow \frac{d}{dx}\left(x\cos(x+y)\right)=\frac{d}{dx}\left(\frac{\pi}{48}\right)$$

Let $u = x$ $\quad V=\cos(x+y)$.

$\frac{du}{dx}=1$ $\quad \frac{dV}{dx}=\frac{d}{dx}\left(\cos(x+y)\right)$. ← Use Chain Rule.

Let $W = x+y$. $\rightarrow V=\cos(W)$.

$\frac{dW}{dx}=1+\frac{dy}{dx}$. $\quad \frac{dV}{dW}=-\sin(W)$.

$$\frac{dV}{dx}=\frac{dV}{dW}\frac{dW}{dx}$$

$$\frac{dV}{dx}=-\sin(x+y)\times\left(1+\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(x\cos(x+y)\right)=\frac{d}{dx}\left(\frac{\pi}{48}\right)$$

$$x\times-\sin(x+y)\left(1+\frac{dy}{dx}\right)+1\times\cos(x+y)=0.$$

at the point $\left(\frac{\pi}{24},\frac{7\pi}{24}\right)$

$$\frac{\pi}{24}\times-\sin\left(\frac{\pi}{24}+\frac{7\pi}{24}\right)\left(1+\frac{dy}{dx}\right)+\cos\left(\frac{\pi}{4}+\frac{7\pi}{4}\right)=0$$

$$-\frac{\pi}{24}\sin\left(\frac{\pi}{3}\right)\left(1+\frac{dy}{dx}\right)+\cos\left(\frac{\pi}{3}\right)=0$$

$$-\frac{\pi}{24}\times\frac{\sqrt{3}}{2}\left(1+\frac{dy}{dx}\right)+\frac{1}{2}=0$$

$$\frac{1}{2}=\frac{\sqrt{3}\,\pi}{48}\left(1+\frac{dy}{dx}\right)$$

$\frac{\pi}{24}+\frac{7\pi}{24}=\frac{8\pi}{24}=\frac{\pi}{3}$

$$\frac{1}{\cancel{2}}\times\frac{\cancel{48}\,24}{\sqrt{3}\,\pi}=1+\frac{dy}{dx}$$

$$\frac{24}{\sqrt{3}\,\pi}\times\frac{\sqrt{3}}{\sqrt{3}}=1+\frac{dy}{dx}$$

$$\frac{8\,\cancel{24}\sqrt{3}}{\cancel{3}\,\pi}=1+\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{8\sqrt{3}}{\pi}-1$$

$$\frac{dy}{dx}=\frac{8\sqrt{3}-\pi}{\pi}$$

Thus the gradient of the curve at $\left(\frac{\pi}{24},\frac{7\pi}{24}\right)$ is $\frac{8\sqrt{3}-\pi}{\pi}$

DO NOT WRITE IN THIS AREA

Question 8 (4 marks)

A body moves in a straight line so that when its displacement from a fixed origin O is x metres, its acceleration, a, is $-4x$ ms^{-2}. The body accelerates from rest and its velocity, v, is equal to -2 ms^{-1} as it passes through the origin. The body then comes to rest again.

$t=0$ $v=0$ $v=-2$ $x=0$.

Find v in terms of x for this interval.

$a = -4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$

$\frac{1}{2}v^2 = -4\int x\,dx$

$\frac{1}{2}v^2 = -4\frac{x^2}{2} + c$

$\frac{1}{2}v^2 = -2x^2 + C.$

$v = -2$ when $x = 0$

$\frac{1}{2}(-2)^2 = -2\times 0^2 + C$

$\frac{1}{2}\times 4 = C.$

$C = 2.$

$\frac{1}{2}v^2 = -2x^2 + 2.$

$v^2 = -4x^2 + 4$

$v = \pm\sqrt{-4x^2+4}$

but $v = -2$ when $x = 0$

$\therefore v = -\sqrt{-4x^2+4}$

or $v = -2\sqrt{-x^2+1}$

or $v = -2\sqrt{1-x^2}$

Question 9 (4 marks)

Given that $f'(x) = \dfrac{\cos(2x)}{\sin^3(2x)}$ and $f\left(\dfrac{\pi}{8}\right) = \dfrac{3}{4}$, find $f(x)$.

$f(x) = \int \frac{\cos(2x)}{\sin^3(2x)}\,dx.$

Let $u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x).$

$\frac{1}{2}\,du = \cos(2x)\,dx.$

$\int \frac{\cos(2x)}{\sin^3(2x)}\,dx = \int \frac{1}{u^3}\times\frac{1}{2}\,du$

$= \frac{1}{2}\int u^{-3}\,du$

$= \frac{1}{2}\frac{u^{-2}}{-2} + c.$

$= -\frac{1}{4u^2} + c.$

$f(x) = -\frac{1}{4\sin^2(2x)} + c.$

$x = \frac{\pi}{8}$ $f\left(\frac{\pi}{8}\right) = \frac{3}{4}$

$\frac{3}{4} = -\frac{1}{4\sin^2\left(\frac{\pi}{4}\right)} + c$

$\frac{3}{4} = -\frac{1}{4\times\left(\frac{\sqrt{2}}{2}\right)^2} + c$

$\frac{3}{4} = -\frac{1}{4\times\frac{2}{4}} + c$

$\frac{3}{4} = -\frac{1}{2} + c$

$c = \frac{5}{4}$

$f(x) = \frac{5}{4} - \frac{1}{4\sin^2(2x)}$

OR $f(x) = \frac{5}{4} - \frac{1}{4}\operatorname{cosec}^2(2x)$

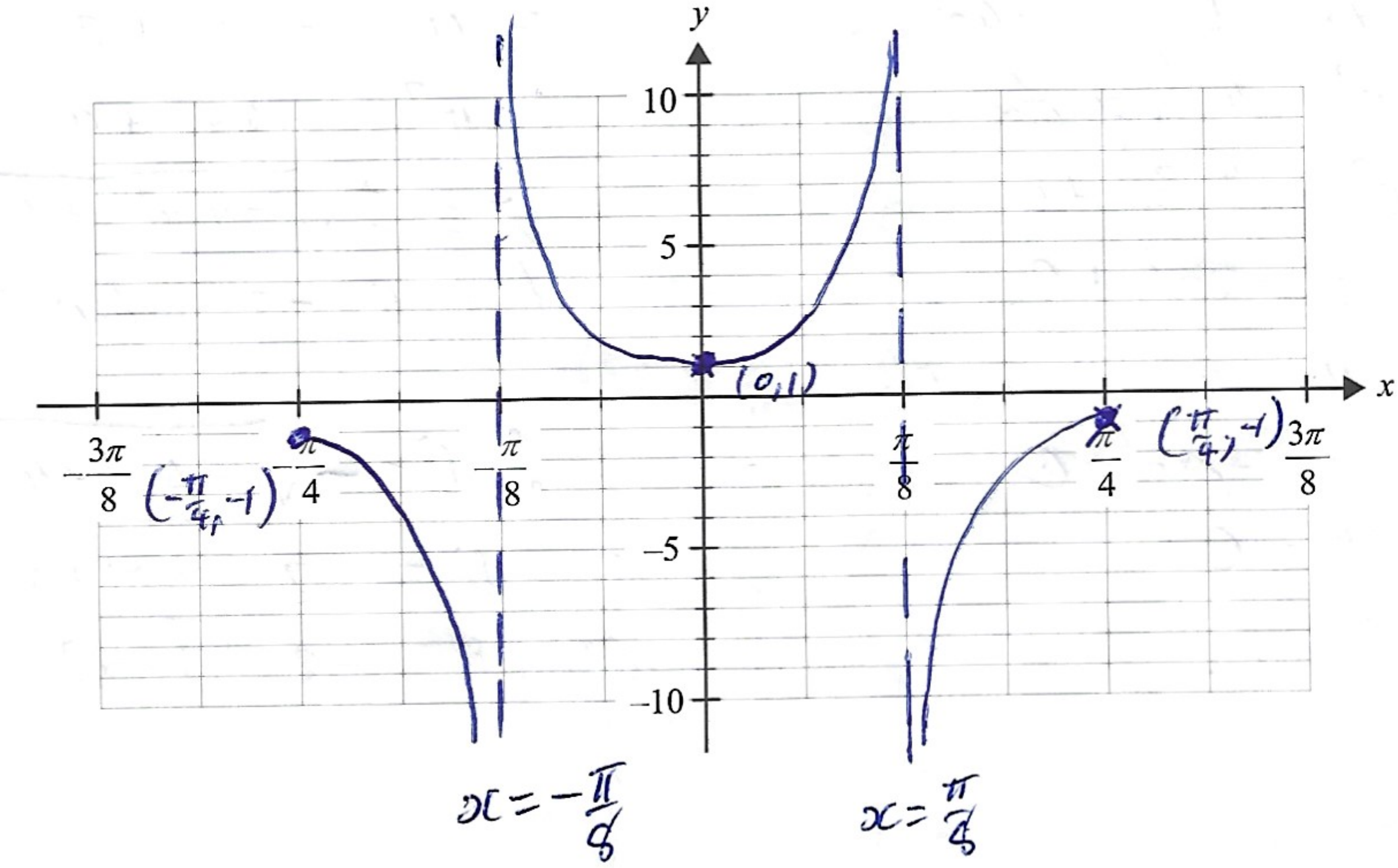
OR $f(x) = \frac{1}{4}\left(5 - \operatorname{cosec}^2(2x)\right)$

TURN OVER

Question 10 (6 marks)

Let $f(x) = \sec(4x)$.

a. Sketch the graph of f for $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ on the set of axes below. Label any asymptotes with their equations and label any turning points and the endpoints with their coordinates. 3 marks



$$\sec(4x) = \frac{1}{\cos(4x)}.$$

$\cos(4x)$ has period $\frac{2\pi}{4} = \frac{\pi}{2}.$

$x = \frac{\pi}{8} \rightarrow 4x = 4 \times \frac{\pi}{8} = \frac{\pi}{2}$

$\cos(\frac{\pi}{2}) = 0.$

Asymptote at $x = \frac{\pi}{8}.$

Also at $x = -\frac{\pi}{8}$

Question 10 – continued

DO NOT WRITE IN THIS AREA

b. The graph of $y = f(x)$ for $x \in \left[-\frac{\pi}{24}, \frac{\pi}{48}\right]$ is rotated about the x-axis to form a solid of revolution.

Find the volume of this solid. Give your answer in the form $\frac{\left(a-\sqrt{b}\right)\pi}{c}$, where $a, b, c \in R$. 3 marks

Note: required form for answer

$$V = \pi \int_{-\frac{\pi}{24}}^{\frac{\pi}{48}} \sec^2(4x)\, dx.$$

$$V = \pi \left[\frac{1}{4} \tan(4x) \right]_{-\frac{\pi}{24}}^{\frac{\pi}{48}}$$

$$= \pi \left[\frac{1}{4} \tan\left(\frac{\pi}{12}\right) - \frac{1}{4} \tan\left(-\frac{\pi}{6}\right) \right]$$

$$= \frac{\pi}{4} \left[\tan\left(\frac{\pi}{12}\right) - \tan\left(-\frac{\pi}{6}\right) \right]$$

$$= \frac{\pi}{4} \left[2-\sqrt{3} \overset{+}{- -} \frac{1}{\sqrt{3}} \right]$$

$$= \frac{\pi}{4} \left[2-\sqrt{3} + \frac{\sqrt{3}}{3} \right]$$

$$= \frac{\pi}{4} \left[\frac{6-3\sqrt{3}+\sqrt{3}}{3} \right]$$

$$= \frac{\pi}{4} \left[\frac{6-2\sqrt{3}}{3} \right]$$

$$= \frac{\pi}{\cancel{4}2} \times \frac{\cancel{2}(3-\sqrt{3})}{3}$$

$$= \frac{(3-\sqrt{3})\pi}{6}.$$

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4} \to \tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

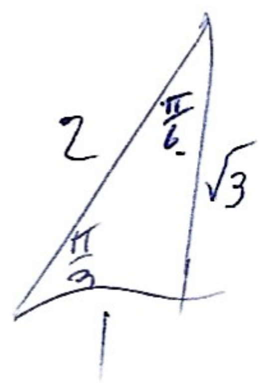
$$= \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}-1)^2}{3-1}$$

$$= \frac{(\sqrt{3}-1)^2}{2}.$$

$$= \frac{3-\sqrt{3}-\sqrt{3}+1}{2} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$



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