VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

CASIO CLASSPAD, USED IN THESE SOLNS

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

SYNNO – SOLNS.

Victorian Certificate of Education 2022

Letter

STUDENT NUMBER

SPECIALIST MATHEMATICS

Written examination 2

Monday 7 November 2022

Reading time: 3.00 pm to 3.15 pm (15 minutes)
Writing time: 3.15 pm to 5.15 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1

For the interval $\frac{1}{2} \le x \le 3$, the graph of $y = |2x-1| - |x-3|$ is the same as the graph of

A. $y = -x - 2$
B. $y = 3x - 4$
C. $y = x + 2$
D. $y = 3x + 2$
E. $y = x - 4$

Graph on Calculator for the domain.
Line has gradient of 3.
B OR D.
Y-intercept is clearly not 2 ⇒ B

Question 2

The expression $1 - \frac{4\sin^2(x)}{\tan^2(x)+1}$ simplifies to

A. $\sin(x)\cos(x)$
B. $1 - 2\cos^2(2x)$
C. $2\sin(2x)$
D. $2\sin^2(2x)$
E. $\cos^2(2x)$

Calculator Simplify gives $\frac{1}{2}(\cos(4x)+1)$
$\frac{1}{2}(2\cos^2(2x) \cancel{-1} \cancel{+1})$
$\frac{1}{2}(2\cos^2(2x))$
$\cos^2(2x)$

Question 3

The graph of $y = \frac{x^2+2x+c}{x^2-4}$, where $c \in R$, will **always** have

A. two vertical asymptotes and one horizontal asymptote.
B. two horizontal asymptotes and one vertical asymptote.
C. a vertical asymptote with equation $x = -2$ and one horizontal asymptote with equation $y = 1$.
D. one horizontal asymptote with equation $y = 1$ and only one vertical asymptote with equation $x = 2$.
E. a horizontal asymptote with equation $y = 1$ and at least one vertical asymptote.

Key Word ALWAYS. → can test. some graphs for different c values.
c = 0 gives 1 vertical $x=2$, 1 horizontal $y=1$
c = 1 gives 2 vert, 1 horizontal.
c = 2 gives 2 vert, 1 horiz.

Question 4

The polynomial $p(z) = (z - a)(z - b)(z - c)$ has complex roots a, b and c, where $\text{Re}(a) \neq 0$, $\text{Re}(b) \neq 0$, $\text{Re}(c) \neq 0$ and $\text{Im}(b) = 0$. When expanded, the polynomial is a cubic with real coefficients.

Which one of the following statements is **necessarily** true?

A. $a + c = 0$

B. $|a| = |c|$

C. $a - c = 0$

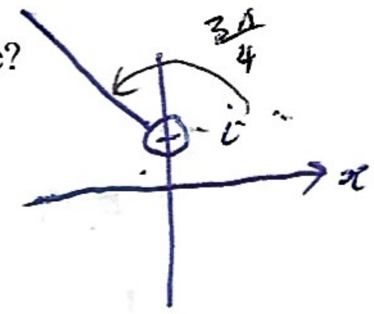
D. $|a| = |b|$

E. $a + b + c = 0$

Question 5

Let $z = x + yi$, where $x, y \in R$ and $z \in C$.

If $\text{Arg}(z - i) = \dfrac{3\pi}{4}$, which one of the following is true?



A. $y = 1 - x, x < 0$

B. $y = 1 - x, x > 0$

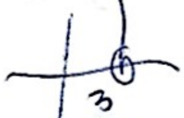
C. $y = 1 + x$

D. $y = 1 + x, x > 0$

E. $y = 1 + x, x < 0$

Question 6

Given $z = x + yi$, where $x, y \in R$ and $z \in C$, an equation that has a graph that has two points of intersection with the graph given by $|z - 5| = 2$ is



A. $\text{Arg}(z - 3) = \dfrac{\pi}{2}$ — No Points

B. $|z - 1| = 2$ — one point $(x-1)^2 + y^2 = 4$

C. $\text{Im}(z) = 2$ — One Point $y = 2$.

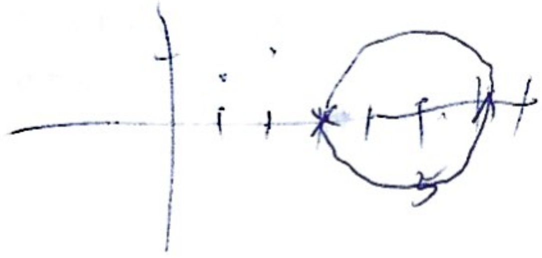
D. $\text{Re}(z) + \text{Im}(z) = 2$ - $x + y = 2 \rightarrow y = 2 - x$ — No Points

E. $|z - 5 - 5i| = 4$

$|z - 5| = 2$.

$\sqrt{(x-5)^2 + y^2} = 2$

$(x-5)^2 + y^2 = 4$



Question 7

Using the substitution $u = 1 + e^x$, $\int_0^{\log_e 2} \frac{1}{1+e^x}\,dx$ can be expressed as

~~A.~~ $\int_0^{\log_e 2}\left(\frac{1}{u-1}-\frac{1}{u}\right)du$

B. $\int_2^{3}\left(\frac{1}{u}-\frac{1}{u-1}\right)du$

~~C.~~ $\int_1^{3}\left(\frac{1}{u}-\frac{1}{u-1}\right)du$

D. $\int_2^{3}\left(\frac{1}{u-1}-\frac{1}{u}\right)du$

~~E.~~ $\int_2^{1+e^2}\left(\frac{1}{u-1}-\frac{1}{u}\right)du$

$u = 1 + e^x$

$x = 0$

$u = 1 + e^0 = 1 + 1 = 2$

$x = \log_e 2$

$u = 1 + e^{\log_e 2} = 1 + 2 = 3$

limits $\int_2^3$ $\rightarrow$ B or D

$\frac{1}{1+e^x} \rightarrow \frac{1}{u}$

$u = 1 + e^x \rightarrow e^x = u - 1$

$\frac{du}{dx} = 0 + e^x = e^x$

$\rightarrow dx = \frac{du}{e^x}$

$= \frac{du}{u-1}$

$\int_2^3 \frac{1}{u} \times \frac{du}{u-1}$

$\int_2^3 \left(\frac{1}{u(u-1)}\right) du$

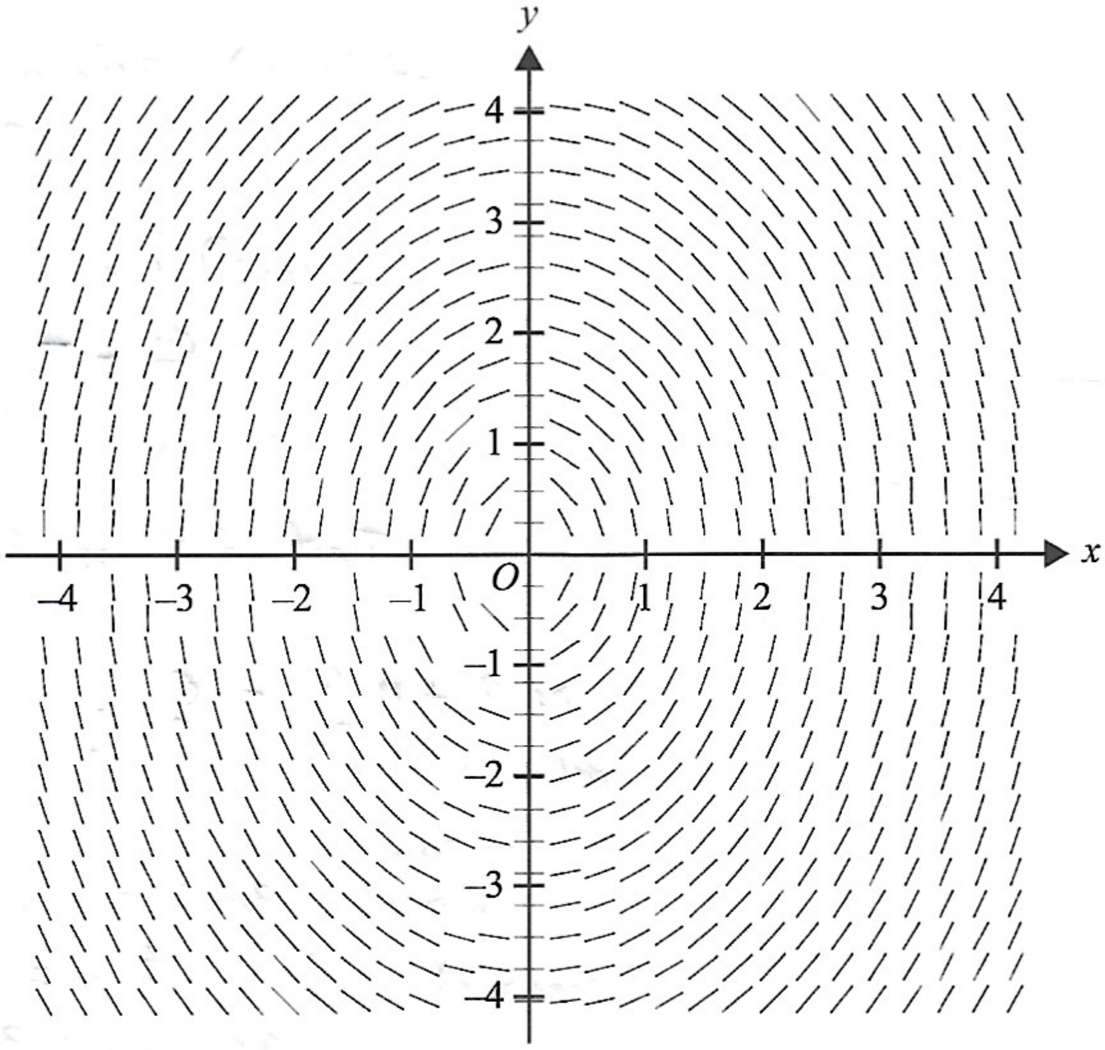
Partial fractions

gives

$\frac{1}{u-1} - \frac{1}{u}$

$\int_2^3 \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

Question 8



The direction field shown above best represents the differential equation

A. $\dfrac{dy}{dx} = \dfrac{2x}{y}$

B. $\dfrac{dy}{dx} = -\dfrac{x}{2y}$

C. $\dfrac{dy}{dx} = -\dfrac{2x}{y}$

D. $\dfrac{dy}{dx} = \dfrac{y^2}{2} + x^2$

E. $\dfrac{dy}{dx} = \dfrac{x^2}{2} + y^2$

Calculator
graph each + compare.
Hint: set window the
same as the graph above.

Question 9

Euler's method is used to find an approximate solution to the differential equation $\dfrac{dy}{dx} = 2x^2$.
Given that $x_0 = 1$, $y_0 = 2$ and $y_2 = 2.976$, the value of the step size h is

A. 0.1

B. 0.2

C. 0.3

D. 0.4

E. 0.5

$x_0 = 1$ $\quad y_0 = 2.$

$x_1 = 1+h$ $\quad y_1 = 2 + h \times 2 \times 1^2$
$= 2 + 2h$

$x_2 = 1+2h$ $\quad y_2 = (2+2h) + h\left(2 \times (1+h)^2\right)$

Calculator SOLVE $\rightarrow 2.976 = (2+2h) + h\left(2(1+h)^2\right)$

Question 10

Consider the curve given by $5x^2y - 3xy + y^2 = 10$.

The equation of the tangent to this curve at the point $(1, m)$, where m is a real constant, will have a negative gradient when

A. $m \in R \setminus [-1, 0]$

B. $m = -\sqrt{11} - 1$ only

C. $m \in R \setminus (-1, 0]$

D. $m = \sqrt{11} - 1$ only

E. $m = -\sqrt{11} - 1$ or $m = \sqrt{11} - 1$

at $(1, m)$ $5 \times 1^2 \times m - 3 \times 1 \times m + m^2 = 10$

$2m + m^2 = 10$

$m = -1 \pm \sqrt{11}$

Calc ImpDiff gives

$\frac{dy}{dx} = -\frac{y(10x-3)}{5x^2 - 3x + 2y}$

at $(1,m)$ $\frac{dy}{dx} = -\frac{7m}{2+2m}$.

Testing Both m above gives −ve gradient

Question 11

Consider the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + p\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - q\underset{\sim}{k}$ and $\underset{\sim}{c} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$, where p and q are real numbers.

If these vectors are linearly **dependent**, then

A. $8p = 5q - 35$

B. $5p = 8q - 35$

C. $8p = -5q - 35$

D. $8p = 5q + 35$

E. $5p = 8q + 35$

$m\underset{\sim}{a} + n\underset{\sim}{b} = \underset{\sim}{c}$.

$m(2i - 3\underset{\sim}{j} + p\underset{\sim}{k}) + n(\underset{\sim}{i} + 2\underset{\sim}{j} - qk) = -3i + 2j + 5k$

i component $2m + n = -3$ } Simultaneous

j component $-3m + 2n = 2$ } $\rightarrow m = -\frac{8}{7}$ $n = -\frac{5}{7}$

$\underset{\sim}{k}$ component $-\frac{8}{7}p + \frac{5}{7}q = 5$

$-8p + 5q = 35$

$5q - 35 = 8p$.

Question 12

Consider the vectors $\underset{\sim}{u}(x) = -\text{cosec}(x)\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$ and $\underset{\sim}{v}(x) = \cos(x)\underset{\sim}{i} + \underset{\sim}{j}$.

If $\underset{\sim}{u}(x)$ is perpendicular to $\underset{\sim}{v}(x)$, then possible values for x are

A. $\frac{\pi}{6}$ and $\frac{7\pi}{6}$

B. $\frac{\pi}{3}$ and $\frac{4\pi}{3}$

C. $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$

D. $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$

E. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

$-\text{cosec}(x) \times \cos(x) + \sqrt{3} \times 1 = 0$

$-\frac{\cos(x)}{\sin(x)} + \sqrt{3} = 0$

$\sqrt{3} = \frac{\cos(x)}{\sin(x)}$

$\frac{1}{\sqrt{3}} = \frac{\sin(x)}{\cos(x)}$.

$\text{Tan}(x) = \frac{1}{\sqrt{3}}$ 1st + 3rd Quad.

$x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$

$= \frac{\pi}{6}, \frac{7\pi}{6}$

Question 13

The acceleration of a body moving in a plane is given by $\ddot{\underset{\sim}{r}}(t) = \sin(t)\underset{\sim}{i} + 2\cos(t)\underset{\sim}{j}$, where $t \geq 0$.

Given that $\dot{\underset{\sim}{r}}(0) = 2\underset{\sim}{i} + \underset{\sim}{j}$, the velocity of the body at time t, $\dot{\underset{\sim}{r}}(t)$, is given by

A. $-\cos(t)\underset{\sim}{i} + 2\sin(t)\underset{\sim}{j}$

B. $(3 - \cos(t))\underset{\sim}{i} + (2\sin(t) + 1)\underset{\sim}{j}$

C. $(1 + \cos(t))\underset{\sim}{i} + (2\sin(t) + 1)\underset{\sim}{j}$

D. $(2 + \sin(t))\underset{\sim}{i} + (2\cos(t) - 1)\underset{\sim}{j}$

E. $(1 + \cos(t))\underset{\sim}{i} + (1 - 2\sin(t))\underset{\sim}{j}$

$\dot{r}(t) = \int \sin(t)\underset{\sim}{i} + 2\cos(t)\underset{\sim}{j}\, dt$

$\dot{r}(t) = -\cos(t)i + 2\sin(t)\underset{\sim}{j} + c$.

$\dot{r}(0) = 2\underset{\sim}{i} + \underset{\sim}{j} = -\cos(0)\underset{\sim}{i} + 2\sin(0)\underset{\sim}{j} + c$

$= -\underset{\sim}{i} + c$.

$c = 3\underset{\sim}{i} + \underset{\sim}{j}$.

$\dot{r}(t) = (3 - \cos(t))\underset{\sim}{i} + (2\sin(t) + 1)\underset{\sim}{j}$

DO NOT WRITE IN THIS AREA

Question 14

A particle moving in a straight line with constant acceleration has a velocity of 7 ms^{-1} at point A and 17 ms^{-1} at point B.

The velocity of the particle, in metres per second, at the midpoint of AB is

A. $\sqrt{119}$

B. 11

C. 12

D. 13

E. $\sqrt{240}$

Note: Displacement $A \to B$ is S

Using $v^2 = u^2 + 2aS$.

$17^2 = 7^2 + 2as$.

$289 = 49 + 2as$.

$as = \frac{289-49}{2} = 120$

at Mid Point Displacement $= \frac{S}{2}$. $v^2 = u^2 + 2a\frac{s}{2}$

$v^2 = 7^2 + as$.

$v^2 = 49 + 120$

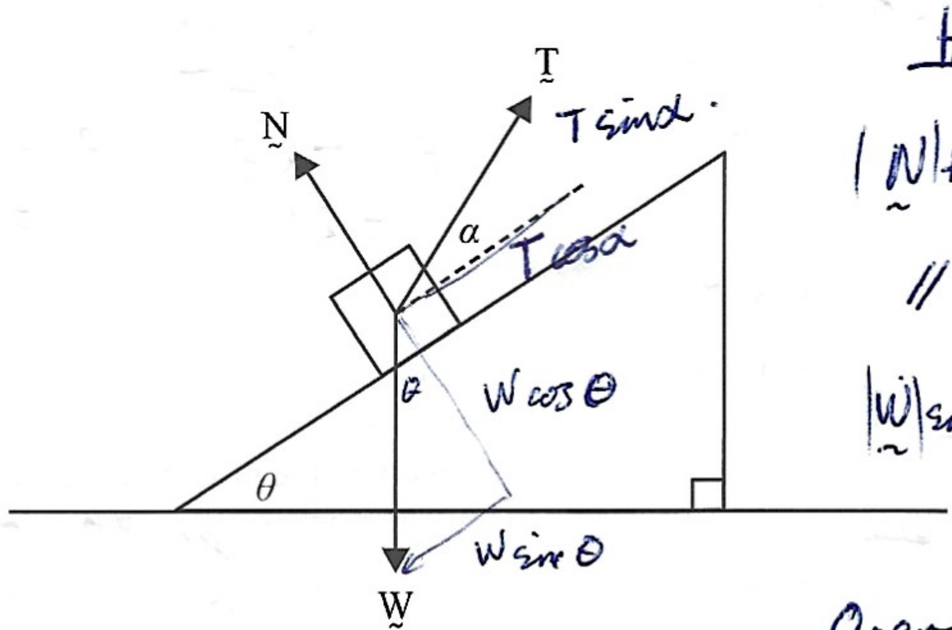
$v^2 = 169$

$v = 13$.

Question 15

An object of mass m kilograms on a smooth inclined plane is acted on by three forces, $\underset{\sim}{N}$, $\underset{\sim}{T}$ and $\underset{\sim}{W}$, as shown in the diagram below. The force $\underset{\sim}{T}$ acts at an angle α up from the slope of the plane. The object experiences an acceleration $\underset{\sim}{a}$.

Eliminates B and D



⊥ to plane. Eliminates A

$|\underset{\sim}{N}| + |\underset{\sim}{T}|\sin\alpha = |\underset{\sim}{W}|\cos\theta$

// to plane (Down Plane +ve)

$|\underset{\sim}{W}|\sin\theta - |\underset{\sim}{T}|\cos\alpha = m|\underset{\sim}{a}|$

Assume of Direction of $\underset{\sim}{a}$ in this.

Overall.

$\underset{\sim}{N} + \underset{\sim}{T} + \underset{\sim}{W} = m\underset{\sim}{a}$.

Which one of the following statements is **necessarily** true?

~~**A.**~~ $|\underset{\sim}{T}|\sin(\alpha) = |\underset{\sim}{W}|\cos(\theta)$

~~**B.**~~ $|\underset{\sim}{W}|\sin(\theta) = |\underset{\sim}{T}|\cos(\alpha)$

C. $|\underset{\sim}{W}|\sin(\theta) - |\underset{\sim}{T}|\cos(\alpha) = m|\underset{\sim}{a}|$

~~**D.**~~ $\underset{\sim}{N} + \underset{\sim}{W} + \underset{\sim}{T} = \underset{\sim}{0}$

E. $\underset{\sim}{N} + \underset{\sim}{W} + \underset{\sim}{T} = m\underset{\sim}{a}$

C ? OR E

Since we don't know the direction of $\underset{\sim}{a}$.

E is the best alternative

C is fine if moving down the plane, but if moving up would be $|\underset{\sim}{T}|\cos(\alpha) - |\underset{\sim}{W}|\sin(\theta) = m|\underset{\sim}{a}|$ since these expressions are using the MAGNITUDE of the vectors.

Question 16

Three coplanar forces of magnitudes 5 N, 7 N and 10 N maintain a particle in equilibrium.

The angle θ between the forces of magnitudes 5 N and 7 N can be found by solving which one of the following equations?

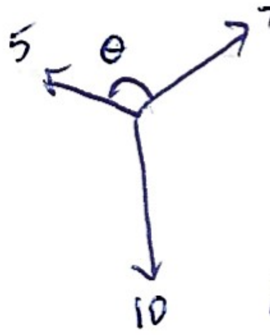
A. $100 = 25 + 49 + 2 \times 5 \times 7 \cos(\theta)$

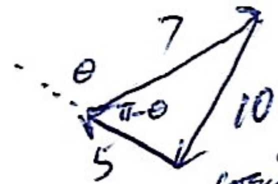
B. $100 = 25 + 49 - 2 \times 5 \times 7 \cos(\theta)$

C. $25 = 100 + 49 - 2 \times 10 \times 7 \cos(\theta)$

D. $49 = 25 + 100 - 2 \times 5 \times 10 \cos(\theta)$

E. $49 = 25 + 100 + 2 \times 5 \times 10 \cos(\theta)$





cosine Rule

$10^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \cos(\pi - \theta)$

but $\cos(\pi - \theta) = -\cos(\theta)$

$100 = 25 + 49 + 2 \times 5 \times 7 \cos(\theta)$

Question 17

A particle of mass 7 kg travels in a straight line with constant acceleration from an initial velocity of 3 ms^{-1}. The particle travels a distance of 30 m in 6 seconds.

The change in momentum of the particle, in kg ms^{-1}, is

A. 0

B. 4

C. 28

D. 49

E. 60

$u = 3. \quad s = 30 \quad t = 6. \quad V = ?$

$s = \frac{1}{2}(u + V)t$

$30 = \frac{1}{2}(3 + V) \times 6$

$10 = 3 + V.$

$V = 7$

$p_i = 7 \times 3$
$= 21$

$p_f = 7 \times 7$
$= 49$

$\Delta p = p_f - p_i$
$= 49 - 21$
$= 28$

Question 18

The time taken, T minutes, for a student to travel to school is normally distributed with a mean of 30 minutes and a standard deviation of 2.5 minutes.

Assuming that individual travel times are independent of each other, the probability, correct to four decimal places, that two consecutive travel times differ by more than 6 minutes is

A. 0.0448

B. 0.0897

C. 0.1151

D. 0.2301

E. 0.9103

$\mu = 30 \quad \sigma = 2.5$

$D = T_1 - T_2.$

$E(D) = E(T_1 - T_2) = 0$

$Pr(|D| > 6) \to 1 - Pr(-6 < D < 6)$
$= 1 - 0.91031397$
$= 0.08968$
$= 0.0897$

$Var(D) = Var(T_1 - T_2)$
$= 1^2 Var(T_1) + (-1)^2 Var(T_2)$
$= Var(T_1) + Var(T_2)$
$= 2.5^2 + 2.5^2$
$= 12.5$

$sd(D) = 3.535533906$

Question 19

The cost, \$$C$, of producing a particular item is a function of its mass, m grams, where $C = 0.3m + 0.5$

The masses of the items are normally distributed with a mean of 7 grams and a standard deviation of 0.1 grams.

$\leftarrow n =$ $\qquad \leftarrow z = 1.96.$

When 100 items are produced, a 95% confidence interval for the average cost per item is closest to

A. (2.502, 2.698)

B. (2.555, 2.645)

C. (2.584, 2.616)

D. (2.594, 2.606)

E. (2.598, 2.602)

$C = 0.3m + 0.5$

$\mu = 7.$

$E(C) = 0.3 \times 7 + 0.5$
$= 2.6$

$sd(m) = 0.1$

$Var(m) = 0.1.$

$Var(C) = Var(0.3 \times m + 0.5)$
$= 0.3^2 Var(m)$
$= 0.0009$

$sd(C) = 0.03.$

$\bar{x} - z \frac{sd}{\sqrt{n}}$

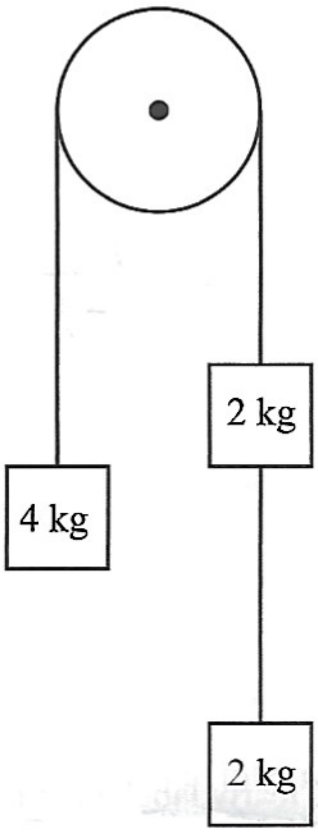
$2.6 - 1.96 \times \frac{0.03}{\sqrt{100}}$

$= 2.59412.$

$= 2.594 \to D$

Question 20

A student constructs the following pulley and mass system using three randomly selected masses that are labelled 4 kg and 2 kg. The masses are connected by two light inextensible strings, one of which passes over a frictionless pulley, as shown below. Initially, the mass labelled 4 kg is held at rest.



If 4 kg moves UP.
then Mass on Right > Mass on left -
Pr(4 kg Moves up) = Pr(Mass Right > Mass left).

$E(R) = 1.980 + 1.980$
$= 3.960$

$Var(R) = Var(2) + Var(2)$
$= 0.015^2 + 0.015^2$
$= 0.00045$

$E(L) = 3.940$

$Var(L) = 0.002^2$

$E(L - R) = 3.94 - 3.96$
$= -0.02$

$Var(L-R) = 1^2 Var(L) + (-1)^2 Var(R)$
$= 0.002^2 + 0.00045$
$= 0.000454$

Most of the system's components are of high quality, but the labels on the masses give only approximations and the actual masses vary, being normally distributed with the following parameters.

Labelled mass (kg)	Mean (kg)	Standard deviation (kg)
2	1.980	0.015
4	3.940	0.002

Correct to three decimal places, the probability that the mass labelled 4 kg moves **up** after it is released is

A. 0.546
B. 0.747
C. 0.826
D. 0.998
E. 1.000

$Pr(L - R) < 0$
$E(L-R) = -0.02$
$sd(L-R) = 0.0213.$
Calc. → 0.826043

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

$x+1+\frac{1}{x-1}$

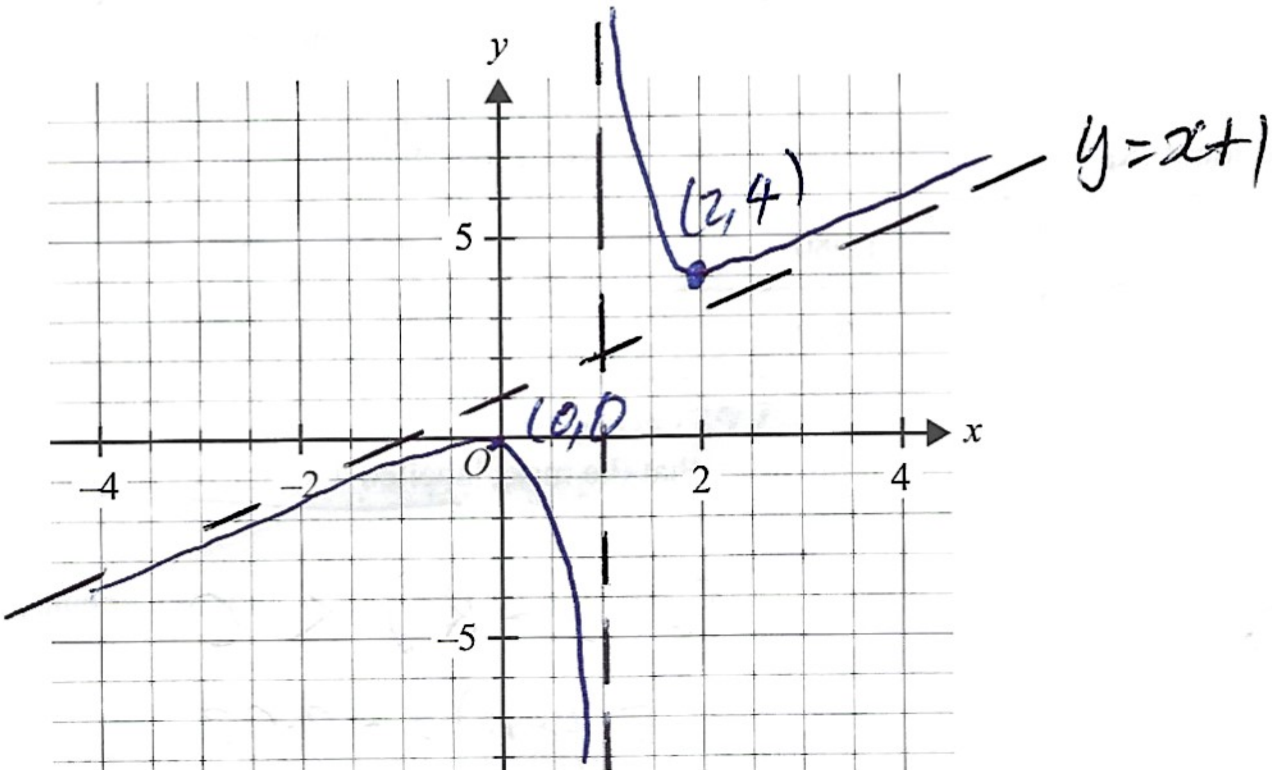
Question 1 (11 marks)

Consider the family of functions f with rule $f(x) = \dfrac{x^2}{x-k}$, where $k \in R\backslash\{0\}$.

a. Write down the equations of the two asymptotes of the graph of f when $k = 1$. 2 marks

$y = x+1$ $\qquad$ $x = 1$

b. Sketch the graph of $y = f(x)$ for $k = 1$ on the set of axes below. Clearly label any turning points with their coordinates and label any asymptotes with their equations. 3 marks



$x = 1$

CAREFUL - The scale on each axis is different

Use calculator to find turning Points

Use a Ruler to draw asymptotes

Related to part a, we have k instead of 1

c. **i.** Find, in terms of k, the equations of the asymptotes of the graph of $f(x)=\dfrac{x^2}{x-k}$. 1 mark

$y = x+k$ $\qquad x = k.$

ii. Find the distance between the two turning points of the graph of $f(x)=\dfrac{x^2}{x-k}$ in terms of k. 2 marks

Relates to Part B. $(2,4) = (2\times1, 4\times1)$

Turning Points $(0,0)$ and $(2k, 4k)$.

distance $= \sqrt{(2k-0)^2+(4k-0)^2}$

$= \sqrt{4k^2+16k^2} = \sqrt{20k^2} = 2\sqrt{5}\,|k|$

d. Now consider the functions h and g, where $h(x) = x + 3$ and $g(x)=\left|\dfrac{x^2}{x-1}\right|$.

The region bounded by the curves of h and g is rotated about the x-axis.

i. Write down the definite integral that can be used to find the volume of the resulting solid. 2 marks

$$\text{Volume} = \pi\int_{\frac{-\sqrt{7}-1}{2}}^{\frac{\sqrt{7}-1}{2}} \left((h(x))^2-(g(x))^2\right)dx.$$

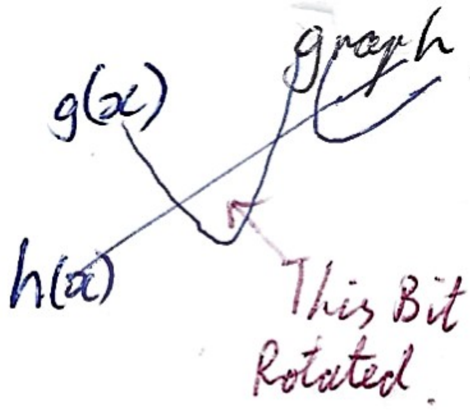
ii. Hence, find the volume of this solid. Give your answer correct to two decimal places. 1 mark

Vol $= 51.41574568$

Volume $= 51.42$.

Use calculator to help for Part d i

graph, Solve $\left(x+3 = \left|\dfrac{x^2}{x-1}\right|\right) \rightarrow \dfrac{3}{2}, \dfrac{-\sqrt{7}-1}{2}, \dfrac{\sqrt{7}-1}{2}$



We want $x = \dfrac{-\sqrt{7}-1}{2}$ and $x = \dfrac{\sqrt{7}-1}{2}$.

$V = \pi\int y^2\,dx$.

$= \pi\int h(x)^2\,dx - \pi\int g(x)^2\,dx$.

$= \pi\int \left[h(x)^2 - g(x)^2\right]dx$.

Question 2 (9 marks) SHOW → Must have all working

Two complex numbers u and v are given by $u = a + i$ and $v = b - \sqrt{2}i$, where $a, b \in R$.

a. **i.** Given that $uv = \left(\sqrt{2} + \sqrt{6}\right) + \left(\sqrt{2} - \sqrt{6}\right)i$, show that $a^2 + \left(1 - \sqrt{3}\right)a - \sqrt{3} = 0$. 2 marks

$uv = (a+i)(b-\sqrt{2}i) = (\sqrt{2}+\sqrt{6}) + (\sqrt{2}-\sqrt{6})i$

$= ab - \sqrt{2}ai + bi - \sqrt{2}i^2$

$= ab + \sqrt{2} + (b - \sqrt{2}a)i$

Real Part Imaginary Part

$ab + \sqrt{2} = \sqrt{2} + \sqrt{6}$ $b - \sqrt{2}a = \sqrt{2} - \sqrt{6}$

$ab = \sqrt{6}$ sub

$b = \dfrac{\sqrt{6}}{a}$

$\dfrac{\sqrt{6}}{a} - \sqrt{2}a = \sqrt{2} - \sqrt{6}$

$\sqrt{6} - \sqrt{2}a^2 = \sqrt{2}a - \sqrt{6}a$

$\cancel{\sqrt{2}}(\sqrt{3} - a^2) = \cancel{\sqrt{2}}a(1 - \sqrt{3})$

$\sqrt{3} - a^2 = a(1 - \sqrt{3})$

$0 = a^2 + (1-\sqrt{3})a - \sqrt{3}$.

ii. One set of possible values for a and b is $a = \sqrt{3}$ and $b = \sqrt{2}$.

Hence, or otherwise, find the other set of possible values. 1 mark

$0 = a^2 + a - \sqrt{3}a - \sqrt{3}$

$0 = a(a+1) - \sqrt{3}(a+1)$

$0 = (a - \sqrt{3})(a+1)$.

$a = \sqrt{3}$ OR $a = -1$.

$b = \dfrac{\sqrt{6}}{a} = -\sqrt{6}$.

$a = -1$, $b = -\sqrt{6}$.

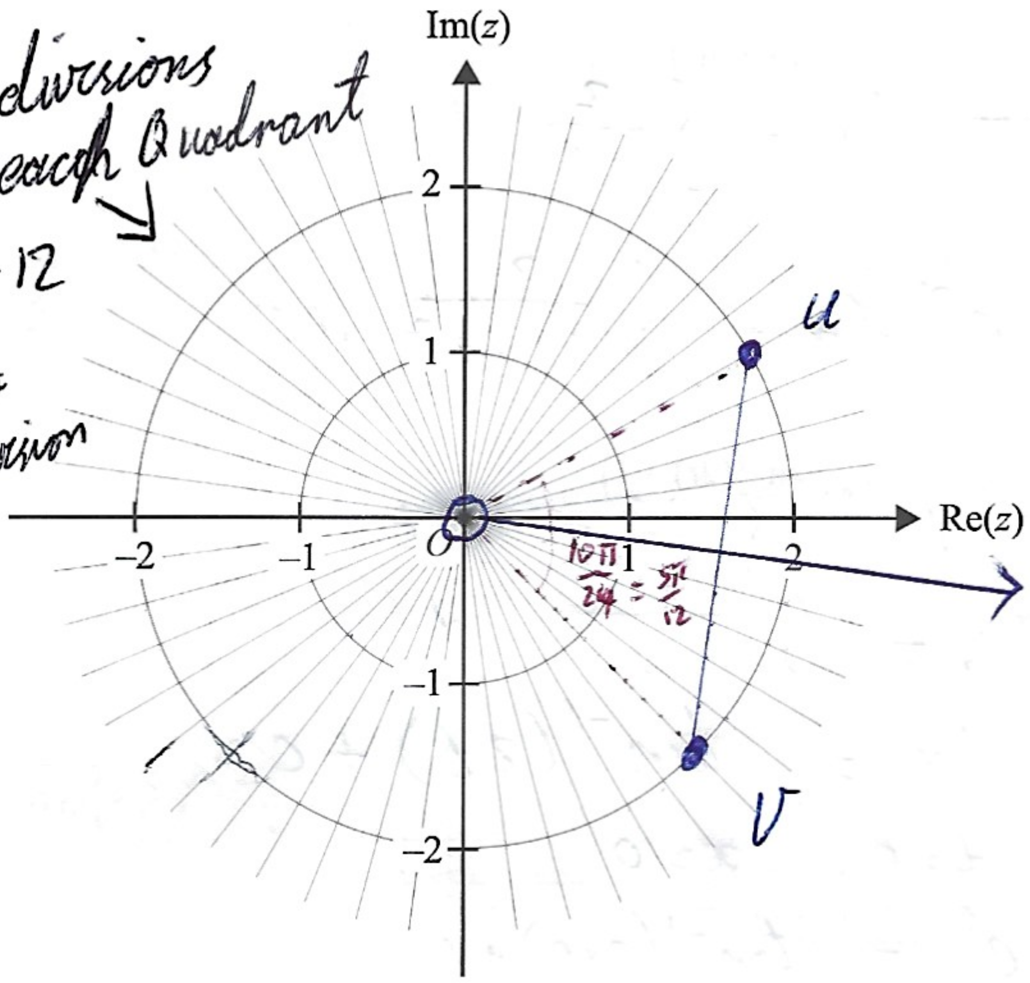
DO NOT WRITE IN THIS AREA

$|u| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2.$ $|v| = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{2+2} = \sqrt{4} = 2.$

b. Plot and label the points representing $u = \sqrt{3} + i$ and $v = \sqrt{2} - \sqrt{2}i$ on the Argand diagram below. 2 marks

Note: 12 divisions in each Quadrant

ie $\frac{\pi}{2} \div 12 = \frac{\pi}{24}$ for each division



c. The ray given by $\text{Arg}(z) = \theta$ passes through the midpoint of the line interval that joins the points $u = \sqrt{3} + i$ and $v = \sqrt{2} - \sqrt{2}i$.

Find, in radians, the value of θ and plot this ray on the Argand diagram in **part b.** 2 marks

Decimal

$\theta = -0.1308996939$

$\frac{\pi}{n} = -0.1308996939$

$n = \frac{\pi}{-0.1308996939} = -24$

Mid Point

$\frac{\sqrt{3}+\sqrt{2}}{2} + \frac{1-\sqrt{2}}{2} i$

$\theta = \tan^{-1}\left(\frac{\frac{1-\sqrt{2}}{2}}{\frac{\sqrt{3}+\sqrt{2}}{2}}\right)$

$\theta = \tan^{-1}\left(\frac{1-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right).$

$\theta = -\frac{\pi}{24}$

d. The line interval that joins the points $u = \sqrt{3} + i$ and $v = \sqrt{2} - \sqrt{2}i$ cuts the circle $|z| = 2$ into a major and a minor segment.

Find the area of the minor segment, giving your answer correct to two decimal places. 2 marks

$A = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} \times 2^2 \times \left(\frac{5\pi}{12} - \sin\left(\frac{5\pi}{12}\right)\right)$

$= 0.6861422254$

$= 0.69.$

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Question 3 (10 marks)

A particle moves in a straight line so that its distance, x metres, from a fixed origin O after time t seconds is given by the differential equation $\frac{dx}{dt} = \frac{2e^{-x}}{1+4t^2}$, where $x = 0$ when $t = 0$.

a. **i.** Express the differential equation in the form $\int g(x)\,dx = \int f(t)\,dt$. 1 mark

$$\int \frac{1}{e^{-x}}\,dx = \int \frac{2}{1+4t^2}\,dt$$

$$\int e^{x}\,dx = \int \frac{2}{1-4t^2}\,dt$$

ii. Hence, show that $x = \log_e\left(\tan^{-1}(2t)+1\right)$. 2 marks

$$\int e^{x}\,dx = \int \frac{2}{1-4t^2}\,dt$$

$$\int e^{x}\,dx = \int \frac{2}{1-(2t)^2}\,dt$$

$$e^{x} = \tan^{-1}(2t) + c$$

$t = 0 \quad x = 0$

$$e^{0} = \tan^{-1}(2\times 0) + c$$

$$1 = \tan^{-1} 0 + c$$

$$1 = 0 + c.$$

$$e^{x} = \tan^{-1}(2t) + 1$$

$$x = \log_e\left(\tan^{-1}(2t)+1\right).$$

b. The graph of $x = \log_e\left(\tan^{-1}(2t) + 1\right)$ has a horizontal asymptote.

i. Write down the equation of this asymptote. 1 mark

$x = \log_e\left(\frac{\pi}{2} + 1\right)$.

$\tan\left(\frac{\pi}{2}\right)$ Undefined $\Rightarrow \tan^{-1}(2t) = \frac{\pi}{2}$ is "undefined" $\Rightarrow$ Asymptote

ii. Sketch the graph of $x = \log_e\left(\tan^{-1}(2t) + 1\right)$ and the horizontal asymptote on the axes below. Using coordinates, plot and label the point where $t = 10$, giving the value of x correct to two decimal places. 2 marks

Asymptote. $x = \log_e\left(\frac{\pi}{2} + 1\right)$
$= 0.9442$
$= 0.94$

$t = 2$
$x = \log_e(\tan^{-1}(4) + 1)$
$= 0.84$

when $t = 10$
$x = \log_e(\tan^{-1}(20) + 1)$
$= 0.9245$
$= 0.92$

when $t = 0$
$x = \log_e(\tan^{-1}(0) + 1)$
$= 0$

c. Find the speed of the particle when $t = 3$. Give your answer in metres per second, correct to two decimal places. 1 mark

$v = \frac{dx}{dt} = \frac{2}{(\tan^{-1}(2t) + 1)(4t^2 + 1)}$

From Calculator, differentiate $\log_e(\tan^{-1}(2t) + 1)$.

$t = 3$.

$v = \frac{2}{(\tan^{-1}(2 \times 3) + 1)(4 \times 3^2 + 1)}$
$= 0.02246$
$= 0.02$

Two seconds after the first particle passed through O, a second particle passes through O. Its distance x metres from O, t seconds after the first particle passed through O, is given by $x = \log_e\left(\tan^{-1}(3t - 6) + 1\right)$.

d. Verify that the particles are the same distance from O when $t = 6$. 1 mark

$t = 6$.
$x = \log_e(\tan^{-1}(2 \times 6) + 1)$
$= 0.9113405378$

$x = \log_e(\tan^{-1}(3 \times 6 - 6) + 1)$
$= 0.9113405378$.

Alternatively Solve $\log_e(\tan^{-1}(2t) + 1) = \log_e(\tan^{-1}(3t - 6) + 1)$
Solution $t = 6$
$\therefore$ same distance from O at $t = 6$.

e. Find the ratio of the speed of the first particle to the speed of the second particle when the particles are at the same distance from O. Give your answer as $\frac{a}{b}$ in simplest form, where a and b are positive integers. ← $t=6$

2 marks

First Particle

$$v_1 = \frac{dx}{dt} = \frac{2}{(\tan^{-1}(2t)+1)(4t^2+1)}$$

$t=6$

$$v_1 = \frac{2}{(\tan^{-1}(2\times6)+1)(4\times6^2+1)}$$

$$= \frac{2}{145\left(-\tan^{-1}\left(\frac{1}{12}\right)+\frac{\pi}{2}+1\right)}$$

Second Particle

$$v_2 = \frac{dx}{dt} = \frac{3}{(\tan^{-1}(3t-6)+1)(9t^2-36t+37)}$$

$t=6$

$$v_2 = \frac{3}{(\tan^{-1}(3\times6-6)+1)(9\times6^2-36\times6+37)}$$

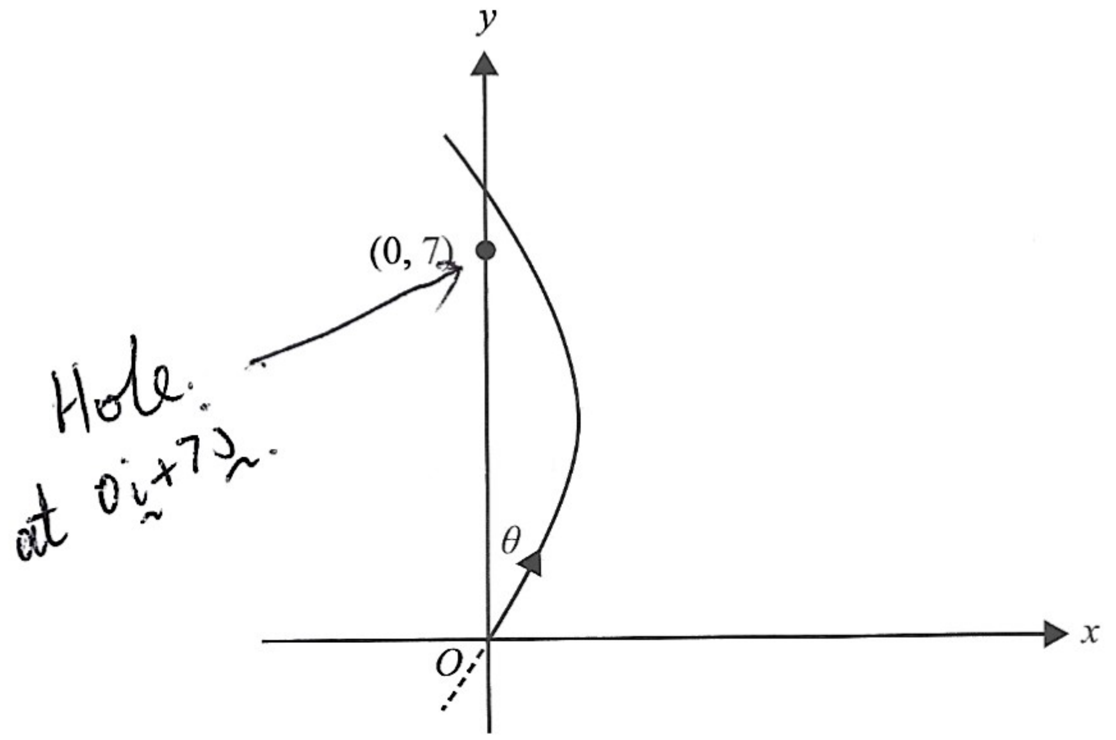
$$= \frac{3}{145\left(-\tan^{-1}\left(\frac{1}{12}\right)+\frac{\pi}{2}+1\right)}$$

$$\frac{v_1}{v_2} = \frac{2}{\cancel{145\left(-\tan^{-1}\left(\frac{1}{12}\right)+\frac{\pi}{2}+1\right)}} \times \frac{\cancel{145\left(-\tan^{-1}\left(\frac{1}{12}\right)+\frac{\pi}{2}+1\right)}}{3}$$

$$= \frac{2}{3}.$$

Question 4 (11 marks)

A student is playing minigolf on a day when there is a very strong wind, which affects the path of the ball. The student hits the ball so that at time $t = 0$ seconds it passes through a fixed origin O. The student aims to hit the ball into a hole that is 7 m from O. When the ball passes through O, its path makes an angle of θ degrees to the forward direction, as shown in the diagram below.



The path of the ball t seconds after passing through O is given by

$$\underset{\sim}{r}(t) = \frac{1}{2}\sin\left(\frac{\pi t}{4}\right)\underset{\sim}{i} + 2t\underset{\sim}{j} \quad \text{for } t \in [0, 5]$$

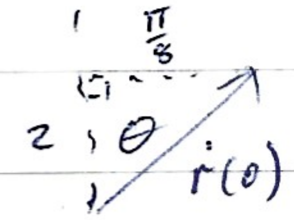
where $\underset{\sim}{i}$ is a unit vector to the right, perpendicular to the forward direction, $\underset{\sim}{j}$ is a unit vector in the forward direction and displacement components are measured in metres.

a. Find θ correct to one decimal place. 2 marks

$\dot{r}(t) = \frac{\pi}{8}\cos\left(\frac{\pi}{4}t\right)\underset{\sim}{i} + 2\underset{\sim}{j}$

$\dot{r}(0) = \frac{\pi}{8}\cos(0)\underset{\sim}{i} + 2\underset{\sim}{j}$

$= \frac{\pi}{8}\underset{\sim}{i} + 2\underset{\sim}{j}$



$\tan\theta = \frac{\pi/8}{2} = \frac{\pi}{8} \times \frac{1}{2} = \frac{\pi}{16}$

$\theta = \tan^{-1}\left(\frac{\pi}{16}\right)$

$= 11.10868$

$= 11.1^{\circ}$

Remember: Velocity includes the direction of travel

θ in Degrees for this Question

b. **i.** Find the speed of the ball as it passes through O. Give your answer in metres per second, correct to two decimal places. 2 marks

$$\text{Speed} = |\dot{r}(0)| = \sqrt{\left(\tfrac{\pi}{8}\right)^2 + 2^2}$$
$$= 2.038188551$$
$$= 2.04$$

ii. Find the minimum speed of the ball, in metres per second, and the time, in seconds, at which this minimum speed occurs. 2 marks

$$\text{Speed} = |\dot{r}(t)| = \sqrt{\left(\tfrac{\pi}{4}\cos\left(\tfrac{\pi}{4}t\right)\right)^2 + 2^2}$$
$$|\dot{r}(2)| = \sqrt{\left(\tfrac{\pi}{8}\cos\left(\tfrac{\pi}{4}\times 2\right)\right)^2 + 2^2}$$
$$= 2.$$

Min Speed when $\ddot{r}(t) = 0$

$$\ddot{r}(t) = -\frac{\pi^2}{32}\sin\left(\tfrac{\pi}{4}t\right)\underset{\sim}{i}$$
$$0 = -\frac{\pi^2}{32}\sin\left(\tfrac{\pi}{4}t\right)$$
$$\sin\left(\tfrac{\pi}{4}t\right) = 0 \rightarrow \tfrac{\pi}{4}t = \tfrac{\pi}{2} \rightarrow t = 2.$$

minSpeed $= 2$ at $t = 2$.

c. Find the minimum distance from the ball to the hole. Give your answer in metres, correct to three decimal places. 3 marks

$$\text{distance} = |\underset{\sim}{r}(t) - 7\underset{\sim}{j}| = \sqrt{\left(\tfrac{1}{2}\sin\left(\tfrac{\pi}{4}t\right)\right)^2 + (2t-7)^2}$$

Min dist where $\frac{d}{dx}|\underset{\sim}{r}(t) - 7\underset{\sim}{j}| = 0$. Calculator Solve.

$$t = 3.516888557.$$

$$\text{Min distance} = \sqrt{\left(\tfrac{1}{2}\sin\left(\tfrac{\pi}{4}\times 3.51\ldots\right)\right)^2 + (2\times 3.51\ldots - 7)^2} = 0.1882527 = 0.188$$

d. How far does the ball travel during the first four seconds after passing through O? Give your answer in metres, correct to three decimal places. 2 marks $t=0 \rightarrow t=4$

$$\text{Distance travelled} = \int_0^4 \sqrt{\left(\tfrac{\pi}{8}\cos\left(\tfrac{\pi}{4}t\right)\right)^2 + 2^2}\, dt$$

$$= 8.076557675$$
$$= 8.077$$

Note:- The path is a curve so distance TRAVELLED is an integral $\int_0^4 \dot{\underset{\sim}{r}}(t)\, dt$.
- Arc length formula on formula sheet.

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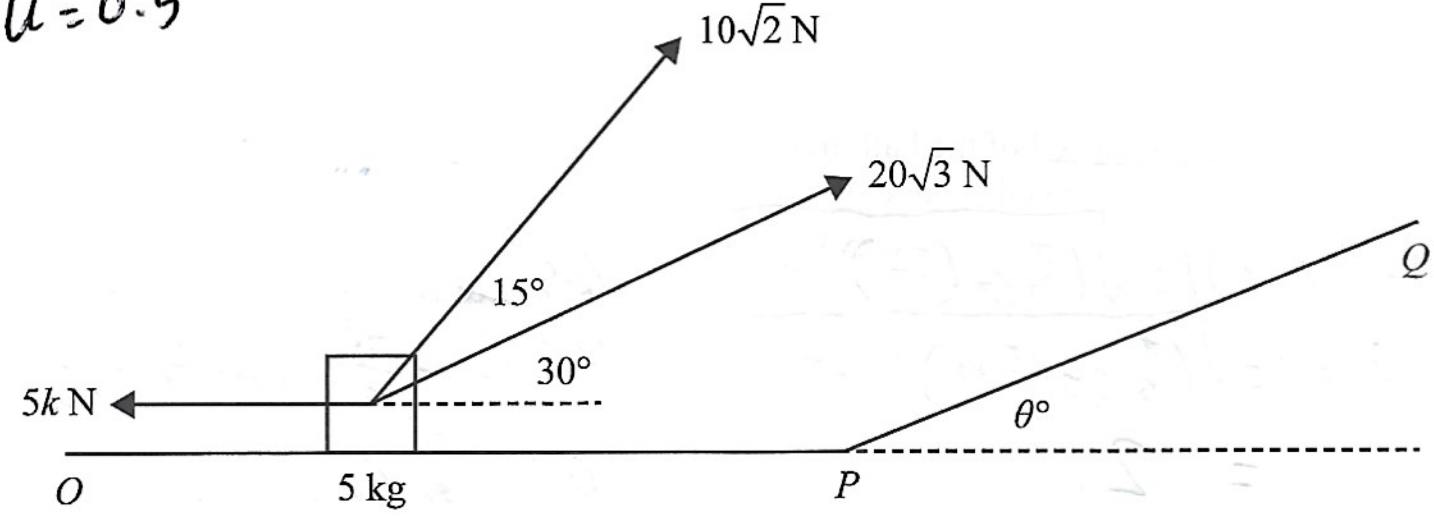
Question 5 (10 marks)

The diagram below shows two forces of magnitudes $10\sqrt{2}$ N and $20\sqrt{3}$ N and a horizontal resistance force of magnitude $5k$ N, where $k \in R$, acting on an object of mass 5 kg. All forces act in the same vertical plane. The force of magnitude $20\sqrt{3}$ N acts at an angle of 30° to the horizontal direction and the force of magnitude $10\sqrt{2}$ N acts at an angle of 15° to the force of magnitude $20\sqrt{3}$ N. The object moves to the right on the horizontal surface and at point O it has a speed of 0.5 ms^{-1}.

$u = 0.5$

Give All Working



a. Show that the acceleration of the object is given by $(8 - k) \text{ ms}^{-2}$. 2 marks

$20\sqrt{3}\cos 30° + 10\sqrt{2}\cos 45° - 5k = 5a$.

$\cancel{20}^{10}\sqrt{3} \times \frac{\sqrt{3}}{\cancel{2}_{1}} + 10\sqrt{2} \times \frac{\sqrt{2}}{2} - 5k = 5a$.

$30 + 10 - 5k = 5a$

$40 - 5k = 5a$

$\cancel{5}(8 - k) = \cancel{5}a$

$a = 8 - k$.

After 5 seconds the object reaches point P, where it has a speed of 2 ms^{-1}. $V = 2$.

b. Find the change in momentum, in kg ms^{-1}, of the object from $t = 0$ to $t = 5$. 1 mark

$\Delta P = P_f - P_i = (mV) - (mu)$

$= (5 \times 2) - (5 \times 0.5)$

$\Delta P = 10 - 2.5$

$= 7.5$

c. Show that $k = 7.7$ 2 marks

$V = u + at$.

$2 = 0.5 + (8 - k) \times 5$

$1.5 = (8 - k) \times 5$

$0.3 = 8 - k$.

$k = 8 - 0.3$

$= 7.7$

Note: $8 - k$ is acceleration $\Rightarrow a = 0.3$

Could use $s = ut + \frac{1}{2}at^2$.

d. Find the distance, in metres, from point O to point P. 2 marks

$v^2 = u^2 + 2as$

$2^2 = 0.5^2 + 2 \times 0.3 \times s$.

$4 = 0.25 + 0.6s$

$3.75 = 0.6s$

$s = 6.25$

e. When the object passes point P, the forces of magnitude $20\sqrt{3}$ N and $10\sqrt{2}$ N cease to act and the object begins to move up the plane PQ at a reduced speed of 1.95 ms^{-1}. The plane is inclined at an angle of $\theta°$ to the horizontal. The object is subject to a resistance force of 38.5 N acting parallel to the plane. The object comes to rest 0.2 m up the plane from P.

$u = 1.95$

Find θ, correct to one decimal place. 3 marks

$v = 0 \quad s = 0.2$

$\underset{\sim}{R} = ma$

$-38.5 - 5 \times 9.8 \sin\theta = 5a$.

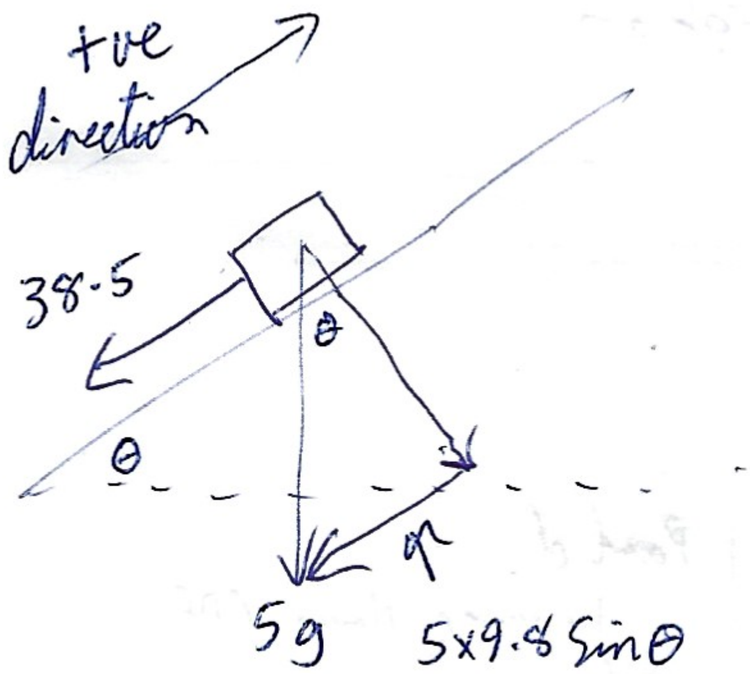
$a = -7.7 - 9.8\sin\theta$

$v^2 = u^2 + 2as$.

$0^2 = 1.95^2 + 2 \times (-7.7 - 9.8\sin\theta) \times 0.2$

$\theta = 10.620977$

$= 10.6°$



Note: use Calculator Solve Function. Careful when interpreting the answer given by the calculator.

Only Friction and gravity now act on the object.

Question 6 (9 marks)

A company produces soft drinks in aluminium cans.

The company sources empty cans from an external supplier, who claims that the mass of aluminium in each can is normally distributed with a mean of 15 grams and a standard deviation of 0.25 grams.

A random sample of 64 empty cans was taken and the mean mass of the sample was found to be 14.94 grams.

Uncertain about the supplier's claim, the company will conduct a one-tailed test at the 5% level of significance. Assume that the standard deviation for the test is 0.25 grams.

a. Write down suitable hypotheses H_0 and H_1 for this test. 1 mark

$H_0 : \mu = 15$

$H_1 : \mu < 15$

In Sample μ was 14.94 Hence <

b. Find the p value for the test, correct to three decimal places. 1 mark

$p = 0.0274289$

$= 0.027$

c. Does the mean mass of the random sample of 64 empty cans support the supplier's claim at the 5% level of significance for a one-tailed test? Justify your answer. 1 mark

P is < 0.05

Thus the claim is not supported.

d. What is the smallest value of the mean mass of the sample of 64 empty cans for H_0 not to be rejected? Give your answer correct to two decimal places. 1 mark

$\bar{x} = 14.94859632$

$= 14.95$

Note: Part b. Using Calculator
STATISTICS → Calc → Test
• Variable
Next
μ condition <
μ_0 15
$\bar{x}$ 14.94
n 25

Part d.
Inverse Norm CDF
Tail Left
Prob 0.05
σ $\frac{0.25}{\sqrt{64}}$
μ 15

SECTION B – Question 6 – continued

The equipment used to package the soft drink weighs each can after the can is filled. It is known from past experience that the masses of cans filled with the soft drink produced by the company are normally distributed with a mean of 406 grams and a standard deviation of 5 grams.

e. What is the probability that the masses of two randomly selected cans of soft drink differ by no more than 3 grams? Give your answer correct to three decimal places. 2 marks

Difference no more than 3g.

$E(C_1 - C_2) = 406 - 406 = 0$

$Var(C_1 - C_2) = 1^2 \times 5^2 + (-1)^2 \times 5^2$

$= 25 + 25$

$= 50$

$sd(C_1 - C_2) = \sqrt{50} = 5\sqrt{2}$.

$Pr(-3 < D < 3)$

$= 0.3286267$

$= 0.329$

f. 1 mL of soft drink has a mass of 1.04 grams. Assume that the empty cans have a mean mass of 15 grams and a standard deviation of 0.25 grams.

What is the probability that a can of soft drink selected at random contains less than 375 mL of soft drink? Give your answer correct to three decimal places. 3 marks

Mass of Soft Drink in full can $= 375 \times 1.04 = 390$

$Pr(\text{Mass} < 390)$.

$E(\text{Mass of Soft Drink}) = E(\text{Full can} - \text{Empty Can}) = 406 - 15 = 391$.

$Var(\text{Mass of Soft Drink}) = Var(\text{Full} - \text{Empty}) = 1^2 \times 5^2 + (-1)^2 \times 0.25^2 = 25.0625$

$\rightarrow Sd(\text{Mass of Drink}) = 5.006246079$

$Pr(\text{Mass} < 390) = 0.42083378207$

$= 0.421$

END OF QUESTION AND ANSWER BOOK

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